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Demand Forecasting for Valuing Design Flexibility

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Abstract. *Scale and scope of demand for long-term infrastructure systems are highly uncertain. Hence, design flexibility that can provide the capability to adapt the infrastructure to unforeseeable changing demand is of value. The use of design flexibility can be justified by determining its economic value using a real options model with Monte Carlo Simulation [1]. The fundamental step towards the use of such a model to value design flexibility is a transformation of forecasting practice, away from aiming to forecast the most likely future demand, and towards forecasting a range of the future demand evolution paths. To this end, we construct a demand forecasting model based on stochastic diffusion processes which are widely used in corporate finance to find the value of a financial option. We also propose an alternative demand forecasting model: adaptive trend fitting with stochastic forecast errors. In our proposed model, we produce one period ahead point forecast by a trend fitting method. Stochastic nature of the point forecast is addressed by a random variable which measures forecast error, i.e., difference between actual and forecasted demand. Monte Carlo Simulation outputs from the two forecasting models are compared.*

Key words*: demand forecast, design flexibility, real options valuation*

1. Introduction

Infrastructure such as buildings, hospitals, bridges, or airports are conventionally built with a fairly rigid design. The systems are largely fixed and irreversible after the completion of construction. Such rigid designs can easily fail when the evolution of the future differs from expectations. This is where flexibility in the infrastructure design can add value. Design flexibility provides the capability to adapt the infrastructure to unforeseeable changing circumstances, as the future unfolds. They have an insurance characteristic. For example, the foundations of a building may be enhanced at the time of initial construction with an extra investment cost, thus providing the flexibility to expand in height at some point in the future if and when demand is larger than originally anticipated. Other examples of design flexibility can be found in the literature, see e.g. [2]-[4].

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Currently, most investments in design flexibility are guided by intuition and experience [5]. However, to make more effective investment decisions, the use of design flexibility needs to be justified by determining its economic value. The application of a real options model to value design flexibility is studied in the literature, see e.g. [1], [5]-[6]. The critical inputs of the real options model are various future uncertain quantities. In this paper, we consider the future demand as the main source of uncertainty that design flexibility is coping with and seek the ways to forecast demand with associated uncertainties.

Current practice in demand forecasting is focused on producing the best possible single projection. To value design flexibility via a real options model, demand forecasts need to exhibit uncertainty and also address the dynamics of the uncertainty. To this end, the objective of this paper is to propose forecasting models that capture stochasticity and dynamicity of future demand. In corporate finance, stochastic diffusion processes are widely used to determine the value of an option on financial security by modeling asset price fluctuations. Thus, one possibility to model future demand dynamics for valuing flexibility would be to treat demand as a random variable in stochastic diffusion processes. Two primary assumptions of this approach are then that demand at time t+1 is dependent only upon demand at time t and the random terms are mutually uncorrelated. In this paper, we propose an alternative demand forecasting method which drops the two principal assumptions of the stochastic diffusion process. Our proposed method produces a deterministic forecast at time t+1 by a trend fitting method. In this way, we express demand at time t+1 as a function of not only demand at time t but also all the previous demand values from time 0 to t. Stochastic nature of the point forecast is addressed by a random variable which measures forecast error, i.e., difference between actual and forecasted demand. These random variables are modeled to be dependent and autocorrelated.

The paper is structured as follows: Chapter 2 defines a demand forecasting problem and notations. In Chapter 3, we model future demand dynamics using stochastic diffusion processes. In Chapter 4, an alternative forecasting model, *adaptive trend fitting with stochastic forecast errors*, is proposed to produce a range of demand paths for valuing design flexibility. Chapter 5 compares the simulation outputs of the two forecasting models. Chapter 6 presents our conclusions.

2. Defining a Demand Forecasting Problem and Notations

A projection of future demand forms an integral part of any design specifications for new infrastructure. Demand forecasting then becomes a crucial task in the design process. Forecasts need to be made for the long-term, covering a substantial part of the lifetime of the infrastructure.

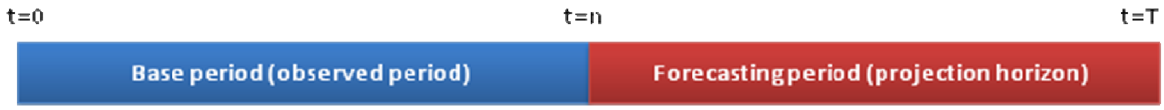

Fig. 1. Timeline of a demand forecasting problem

We define our demand forecasting problem by setting up two periods as shown in Fig. 1: base period from time 0 to n during when we have the observed data and forecasting period from time n+1 to T during when we forecast future demand. T represents the end of lifetime of a new infrastructure.

Next we define some notation for time variables:

t	time variable, t = 0,1,2,…,n,…,T
$[t_1, t_2]$	time period between t_1 and t_2
$[0, t_0]$	warm-up period
$[0, n]$	base period
$[n+1, T]$	forecasting period

We aim to model dynamicity and stochasticity of future demand by generating a range of future demand paths by Monte Carlo Simulation. The use of simulation encoding of the mathematics representation of ensuing demand, for example in a spreadsheet for real options valuation in [1] is more applicable in the practical design process. We propose simulation-based forecasting models for valuing design flexibility in the next two chapters.

3. **Demand Forecasting Model using Stochastic Diffusion Processes**

Stochastic diffusion processes are widely used to model asset dynamics in corporate finance. We construct a demand forecasting model based on stochastic diffusion processes to value design flexibility using a real options model. If we represent stochastic diffusion processes as a discrete time model, the development of stochastic demand, y_t during the projection period can be modeled recursively as [7]:

$$y_{t+1} = y_t u_t \tag{1}$$

for t=[n+1,T]. The quantities u_t are mutually independent random variables. The variable u_t defines the relative change in demand between t and t+1. The relative change will be independent of the overall magnitude of y_t. Since the demand will never become negative, we consider the logarithm of demand. If we take the natural logarithm of both sides of the equation, the equation is equivalent to

$$\ln y_{t+1} = \ln y_t + \ln u_t \tag{2}$$

for t=[n+1,T]. If we introduce the random disturbances directly in terms of the $\ln u_t$,

$$w_t = \ln u_t \tag{3}$$

for t=[n+1,T]. We let these w_t's be mutually independent normal random variables with the following mean and variance using the actual demand data after a predefined warm-up period:

$$\mu = E[w_t] = E[\ln y_{t+1} - \ln y_t] \tag{4}$$

$$\sigma^2 = \mathrm{var}[w_t] = \mathrm{var}[\ln y_{t+1} - \ln y_t]. \tag{5}$$

for t=[t_0,n]. The mean μ is referred as the drift term which describes the trend in the variable, and the variance σ^2 is referred as the volatility term. Eq. (2) can be written in terms of the drift and the volatility term as:

$$\ln y_{t+1} = \ln y_t + \mu + \sigma\varepsilon_t \tag{6}$$

The random variable in Eq. (6) is ε_t. In Monte Carlo Simulation, we randomly choose a value from an assumed distribution, typically a normal distribution, for this variable to obtain a possible value or a simulated value of $\ln y_{t+1}$ in Eq. (6). By repeating the same procedure, a range of demand scenarios are generated (Fig. 2).

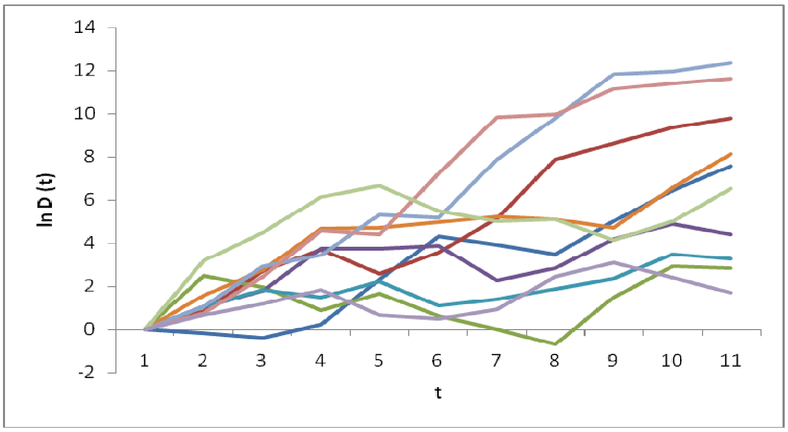

Fig. 2. Ten demand scenarios generated by Monte Carlo Simulation with an assumption of stochastic diffusion processes. The case of y(0)=1, μ=0.5, σ^2=1.

4. Alternative Demand Forecasting Model: Adaptive Trend Fitting with Stochastic Forecast Errors

The demand forecasting model developed using stochastic diffusion processes in Chapter 2 make two primary assumptions: 1. Demand at time t+1 is dependent only upon demand at time t, 2. The random terms are mutually uncorrelated. However, these assumptions are unrealistic for the demand process which does not have a Markov chain property. We propose an alternative demand forecasting model, *an adaptive trend fitting with stochastic forecast errors* which drops the two principal assumptions of the previous model. Our proposed model produces a deterministic point forecast at time t+1 by a trend fitting method. A trend fitting method fits trend curves, mostly linear, exponential, or parabolic curves, to the observed data and extrapolates the trend into the future. In this way, we express demand at time t+1 as a function of not only demand at time t but also all the previous demand values from the period [0,t]. Stochastic nature of the point forecast is addressed by a random variable which measures forecast error, i.e., difference between actual and forecasted demand. These random variables are modeled to be dependent and autocorrelated. The underlying reason is that recurrent forecasts are made by the same trend fitting method and share historical data. The development of stochastic demand, y_t at time t can then be modeled recursively as

$$y_{t+1} = f(t; y_0, y_1, ..., y_t) + E_{t+1} \tag{7}$$

for t=[n+1,T]. The function, f represents a trend fitting and it outputs a deterministic point forecast at time t+1 using the demand data from the period [0,t]. E_{t+1} is a random variable which accounts for an unknown forecast error of a point forecast made by a trend fitting. Monte Carlo simulation is used to sample future forecast error from the assumed distribution. By iteratively generating a new deterministic forecast and adding a forecast error as in Fig. 3, we can map a simulated path of demand. We repeat the process for many simulations and generate a range of demand scenarios.

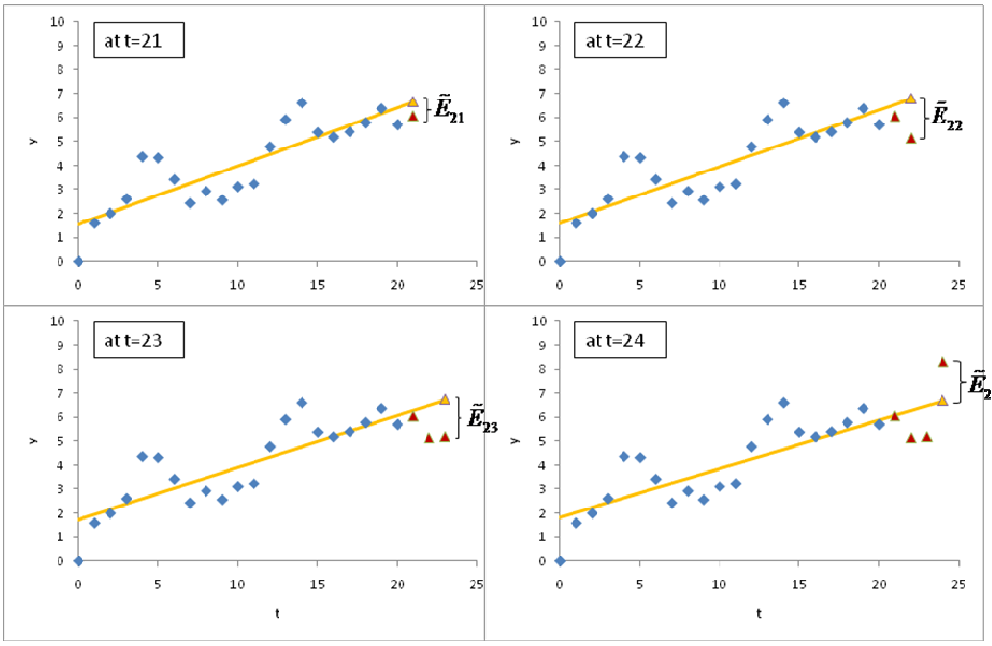

Fig. 3. Graphical illustration of an adaptive trend fitting with stochastic forecast errors: the case of a linear fitting. A deterministic forecast at t=21 is made by fitting a linear line to observed data for the period [0,20], represented by a yellow triangle. A simulated forecast error, $\tilde{E}_{21}$ is then added to the deterministic forecast to correct a point forecast to a simulated actual, represented by a red triangle. The model repeats the process to map a simulated path of future demand.

5. Comparing the Two Forecasting Models

We illustrate the performance of the adaptive trend fitting with stochastic forecast errors in Chapter 4, which is hereby referred as *forecaster II,* in comparison to the forecasting model developed using stochastic diffusion processes in Chapter 3, which is hereby referred as *forecaster I*. We assume that demand follows an additive random walk, a simplified *forecaster I*, i.e. actual demand can be forecasted using *forecaster I*. We then evaluate the performance of *forecaster II* for forecasting this demand process using Monte Carlo Simulation. In this way, we are setting *forecaster I* as a reference model for a comparison with *forecaster II.*

Forecaster I:

We simplify *forecaster I* by assigning zero to the drift term and one to the volatility term, i.e. $\mu = 0, \sigma = 1$ in Eq. (6). If we let a fundamental variable as y_t and assume this variable can explain the actual demand process, actual demand is explained by:

$$y_{t+1} = y_t + \varepsilon_t \tag{8}$$

for t=[0,T]. We further simplify Eq. (8) by assuming that ε_t is independent and identically distributed and has a normal distribution with zero mean and $\sigma_\varepsilon{}^2$ variance, i.e., $\varepsilon_t \sim N(0, \sigma_\varepsilon{}^2)$. We also assume that the initial demand is zero, i.e., $y_0 = 0$. Then Eq.(8) is equivalent to

$$y_{t+1} = \sum_{i=0}^{t} \varepsilon_i \tag{9}$$

for t=[0,T]. Namely, stochastic demand at time t+1 is a sum of independently simulated normal random variables, ε_i for 0≤i≤t.

Forecaster II:

We model the assumed demand process described by Eq. (9) by *forecaster II*. We express forecasted demand $\tilde{y}_t$ by *forecaster II* as*:*

$$\tilde{y}_{t+1} = f(t; y_0, y_1, ..., y_t) + E_{t+1} \tag{10}$$

for t=[n+1,T].

We consider a linear trend fitting as an underlying demand model to produce a deterministic forecast. The function, *f* for a linear trend fit then becomes:

$$f(t; y_0, y_1, ..., y_t) = a_t + b_t t \tag{11}$$

where a_t and b_t are parameters calculated using available demand data over the period [0,t] and have closed form solutions as:

$$a_t = \bar{y} - \bar{t}\,b_t = \frac{\sum_{i=0}^{t} y_i}{t+1} - \frac{t}{2} b_t \tag{12}$$

$$b_t = \frac{\sum_{i=0}^{t} (t_i - \bar{t})(y_i - \bar{y})}{\sum_{i=0}^{t} (t_i - \bar{t})^2} = \frac{\sum_{i=0}^{t} (t_i - \frac{t}{2}) y_i}{\sum_{i=0}^{t} (t_i - \frac{t}{2})^2} \tag{13}$$

We rewrite Eq.(10) in terms of these parameters.

$$\tilde{y}_{t+1} = a_t + b_t (t+1) + E_{t+1} \tag{14}$$

for t=[n+1,T].

The key question is then how to sample appropriate forecast errors, E_{t+1} over projection horizons, i.e. t=[n+1,T]. We assume that a distribution of future forecast errors can be obtained through an analysis of historical forecast errors. To estimate a realistic distribution of future forecast errors, we need a set of historical forecast errors. However, data for historical forecast errors are often not systematically collected or not accessible in practice. We therefore generate these error terms empirically by generating predictions of the forecast at times t=0,…,t=n by the chosen trend fitting method for forecasting, related to the observed data during the base period after a predefined warm-up period, t=[0,t_0]. We hereby define empirically collected forecast error at time t as historical forecast error, e_t. A detailed study on the characteristics of historical forecast error and the assumptions on the distribution of future forecast error based on these characteristics is referred to Appendix A and Appendix B.

Comparing Forecaster I with Forecaster II

Next we illustrate *forecaster I* described by Eq. (9) and *forecaster II* described by Eq. (14) via Monte Carlo Simulation. Fig. 4 shows the distributions that are generated by *forecaster I* and *forecaster II* at t=60 using the base period of [0,40] (i.e. n=40). This figure shows that *forecaster II* produces an unbiased estimate of *forecaster I* and a higher variance. *Forecaster II* results this higher variance since it accounts for not only the variability in the assumed demand time series that are modeled by *forecaster I* but also errors in the chosen trend fitting method. Both time series variability and model fitting error are captured in *forecaster II* by introducing a random variable that measures a difference between actual and forecasted demand.

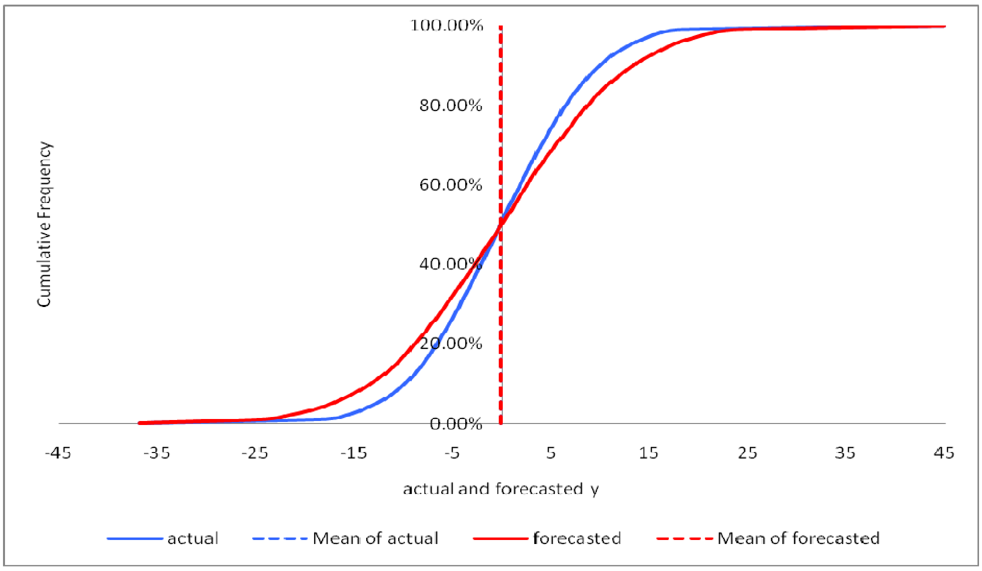

Fig. 4. Cumulative distributions of the actual demand simulated by *forecaster I* and forecasted demand simulated by *forecaster II* at t=60 using the base period [0,40]**.** Simulation result is based on 10,000 Monte Carlo trials.

6. Conclusion

Scale and scope of demand for long-term infrastructure systems are highly uncertain. Hence, design flexibility which provides adaptability of such systems to changing circumstances is of value. To make the investment in design flexibility more effective, the economic value of design flexibility needs to be determined using a real options model. Traditional demand forecasting methodologies, whilst allowing for the estimation of prediction errors at any point in time, are not set up to produce fully dynamic stochastic models that are needed for real options valuation. In this paper, we develop two simulation-based demand forecasting models that can produce a range of future demand paths as a representation of dynamicity and stochasticity of future demand uncertainty.

We construct *forecaster I* by transforming stochastic diffusion processes in corporate finance that are conventionally used to price a financial option. We develop an alternative forecasting model: *adaptive trend fitting with stochastic forecast errors* and this model is referred as *forecaster II.* At time t, *forecaster II* produces a point forecast at time t+1 and adds a stochastic forecast error term, and proceeds in this way to t+1,….,T. The random variable that measures forecast error produces the dynamic stochastic nature of this forecast. This random variable can model both, variability in the demand time series and errors in the forecasting model unlike *forecaster I* which accounts for time series variability alone.

Distributions obtained from Monte Carlo Simulation suggest that *forecaster II* produces an unbiased estimate of *forecaster I.* Thus the performance of *forecaster II* for forecasting the mean of stochastic demand process is equal to that of *forecaster I.* In addition, *forecaster II* produces a higher variance. This higher variability is what we should include in a real options model for valuing design flexibility when actual demand process cannot be described by stochastic diffusion processes. Markov chain assumption in stochastic diffusion processes can often be unrealistic to describe actual demand time series. In this case, *forecaster II* is more general in a way that it accounts for both time series variability and model fitting error. Forecasting models based on statistical estimates of time series variability alone like *forecaster I* will overestimate our knowledge and underestimate the uncertainty.

Future research direction will be applying the two forecasters to real demand data.

7. Acknowledgement

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8. References

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Appendix A: Analysis of Historical Forecast Errors (0≤t≤n)

Actual demand y is assumed to follow a simplified stochastic diffusion process, namely an additive random walk in Chapter 5, i.e. actual demand can be exactly modeled by *forecaster I*. Actual demand can then be described as a sum of i.i.d normal random variables, $\varepsilon \sim N(0, {\sigma_\varepsilon}^2)$ when initial demand is assumed to be zero:

$$y_{t+1} = \sum_{i=0}^{t} \varepsilon_i \qquad \text{(A1)}$$

for t=[0,T].

In Chapter 5, for simplicity, we choose a linear trend fitting as an underlying demand model to produce a deterministic forecast in *Forecaster II*. To assume a distribution of future forecast errors, denoted by E_{t+1}, we first analyze the characteristics of historical forecast errors, denoted by e_t. We collect historical forecast error terms empirically by generating predictions of the forecast at times t=0,…,t=n by the chosen linear trend fitting method for forecasting, related to the observed data at hand.

The fitted demand $\hat{y}_t$ by a linear trending is

$$\hat{y}_{t+1} = f(t+1; y_0, y_1, ..., y_t) = a_t + b_t(t+1) \qquad \text{(A2)}$$

for t=[t$_0$,n]. a_t and b_t are parameters calculated using available demand data over the period [0,t] and have closed form solutions as:

$$a_t = \bar{y} - \bar{t}\, b_t = \frac{\sum_{i=0}^{t} y_i}{t+1} - \frac{t}{2} b_t \qquad \text{(A3)}$$

$$b_t = \frac{\sum_{i=0}^{t} (t_i - \bar{t})(y_i - \bar{y})}{\sum_{i=0}^{t} (t_i - \bar{t})^2} = \frac{\sum_{i=0}^{t} (t_i - \frac{t}{2}) y_i}{\sum_{i=0}^{t} (t_i - \frac{t}{2})^2} \qquad \text{(A4)}$$

We can express the fitted demand as linear functions of the same normal random variables, ε_i in Eq. (A1).

$$\hat{y}_{t+1} = \sum_{i=0}^{t-1} \varepsilon_i B_i^t \qquad \text{(A5)}$$

for t=[t$_0$,n]. B_i^t is a coefficient of ith normal random variable, ε_i where 0≤i≤t-1 to find the fitted value at time t+1 (see Appendix C and D for details). Since $\hat{y}_{t+1}$ is a linear function of independent normal random variables, it is also normally distributed.

The generated historical forecast errors are the differences between the actual and the predicted values. Eq. (A1) and Eq. (A5) therefore yield the following expression for these errors:

$$e_{t+1} = y_{t+1} - \hat{y}_{t+1} = \sum_{i=0}^{t} \varepsilon_i - \sum_{i=0}^{t-1} \varepsilon_i B_i^t = \sum_{i=0}^{t-1} \varepsilon_i (1 - B_i^t) + \varepsilon_t \tag{A6}$$

where $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$. Historical forecast error terms are collected over the base period after a warm-up period, i.e. t=[t$_0$,n].

Given known values, $y_0, y_1, ..., y_t$, or given known values, $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{t-1}$, the linear fitting method assigns different weights to ε_i terms and sum them up to make a prediction at time t+1 (Eq.(A5)). On the other hand, the actual demand at time t+1 is the sum of $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{t-1}, \varepsilon_t$ with an equal weight attached to each term (Eq.(A1)). Note there is one more random variable, ε_t in the expression of the actual demand. Therefore, the sums of the differences in the assigned weights to ε_i terms in the actual demand and the fitted demand account for the magnitude of historical forecast error. Fig. A1 illustrates the differences in the assigned weights or coefficients of ε_i for the actual and fitted demand at time t=20.

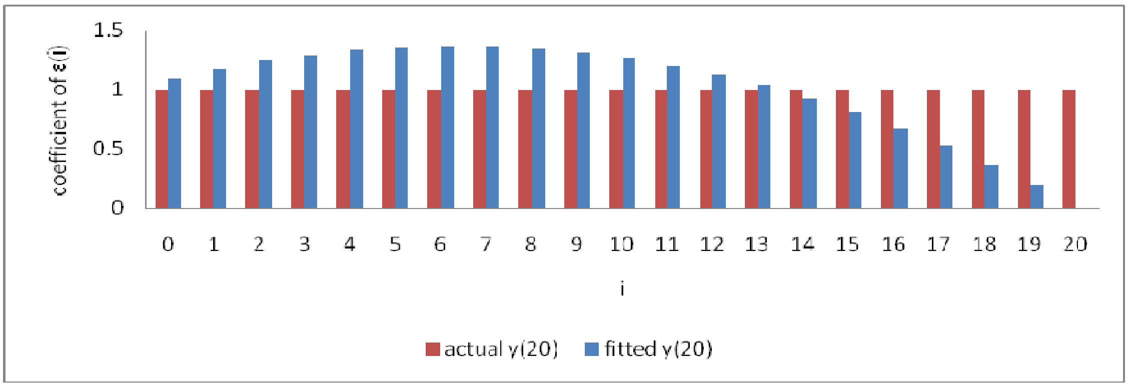

Fig. A1. Coefficients of ε_i where 0≤i≤20 for the actual and fitted demand at t=20

We will next discuss some of the characteristics of historical forecast errors.

Characteristic 1

It is illustrative to decompose Eq.(A6) into its leading and trailing term. The leading term finds the error due to the differences in the weights of ε_i terms from i=0 to i=t-1 and these differences arise due to errors associated with the linear model fitting. The trailing term explain the realized variability of the actual demand from time t to time t+1.

$$e_{t+1} = \underbrace{\sum_{i=0}^{t-1} \varepsilon_i (1 - B_i^t)}_{\text{model fitting error}} + \overbrace{\varepsilon_t}^{\text{time series variability}} \qquad \text{(A6)-1}$$

Characteristic 2

The forecast error is a linear function of independent normal random variables and therefore it is also normally distributed.

Characteristic 3

The expected value of forecast error is zero.

$$E\left[e_{t+1}\right] = E\left[\sum_{i=0}^{t-1} \varepsilon_i (1 - B_i^t)\right] + E\left[\varepsilon_t\right] = 0 \qquad \text{(A7)}$$

Characteristic 4

The variance of the historical forecast error at time t+1 is

$$\sigma_{e_{t+1}}^2 = \underbrace{\sum_{i=0}^{t-1} (1 - B_i^t)^2 \sigma_\varepsilon^2}_{\text{variance of model fitting error}} + \overbrace{\sigma_\varepsilon^2}^{\text{variance of time series variability}} \qquad \text{(A8)}$$

We also decompose Eq.(A8) into its leading and trailing term. The leading term expresses the variability of the model fitting error. This is a function of B_i^t whose values vary with time. Hence, the variance of the model fitting error will not be constant over time. In fact, the variability increases as time increases (Fig. A2). This is somewhat counterintuitive. One would expect the variability of the model error to reduce over time as more data becomes available. The increase in the model fitting error variance is due to the characteristic of the assumed time series, which adds a random increment to the previous time series. The random increments will accumulate, i.e. the time series itself becomes more difficult to predict over time. This out-weighs the advantage of additional data.

The trailing term is the variance of time series variability between time t and time t+1 and this is constant over time in a random walk time series (Fig. A2).

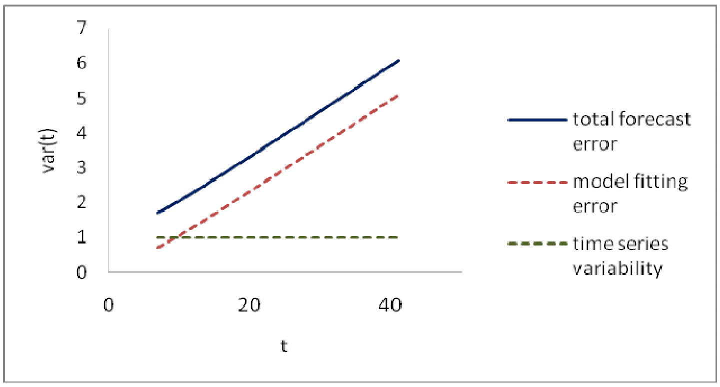

Fig. A2. Variance of errors when $\sigma_\varepsilon^2=1$ for $5\leq t\leq 40$ (the case of $t_0=4$, $n=40$)

Appendix B: Distribution of Future Forecast Errors (n≤t≤T)

We show that the historical forecast errors are normal random variables with zero expectation in Appendix A. But, they are not independent and not identically distributed as the variance of forecast errors is not fixed but increases over time. Therefore we cannot simulate future forecast errors by sampling directly from historical forecast errors as if they were i.i.d random variables. We will have to deal with the non-stationary characteristics of the error term process.

For simulation purposes, we assume that the distribution of future forecast errors has three characteristics: First, future forecast errors are normally distributed. Second, the distribution has expected value zero. Third, the variance of future forecast error will increase at the same rate as in the past. Fig. A2 illustrates that assumption of a linear variance growth is justified in the case of a random walk time series. To calculate the rate of growth of the variance of future forecast errors we use the average of historical growth rates.

The historical forecast errors are autocorrelated. The underlying reason is that recurrent forecasts rely on the same general method and share historical data. Therefore, we need to account for the autocorrelation when we sample future forecast errors. We build a first order autoregressive (AR(1)) model of future forecast errors to address the lag 1 autocorrelation. Given the historical forecast error at t=n, we model the future forecast error at t=n+1 by a AR(1) model and repeat the process. The details of the model are given in Appendix E.

Appendix C: The Coefficient, B_i^t in the Expression of the Fitted Demand, $\hat{y}_t$

The coefficient, B_i^t in Eq. (A5) is a function of both i and t i.e., the value of the coefficient varies with i as well as t. We represent the coefficient values when t=10 and t=40 to illustrate how the value varies with i at a given time in Fig. A3.

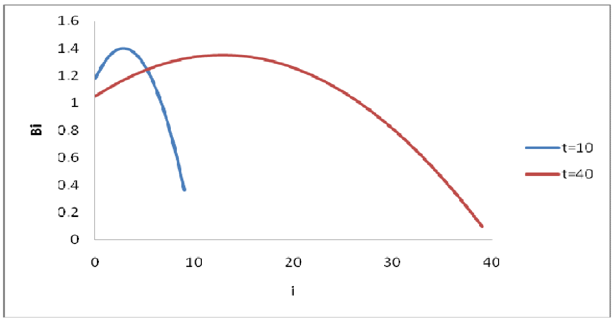

Fig. A3. Values of B_i when t=10 and t=40

Appendix D: The Derivation of Eq. (A5)

This appendix provides the derivation of the fitted demand at time t+1, $\hat{y}_{t+1}$ as a linear function of i.i.d. normal random variables, $\varepsilon \sim N(0, \sigma_\varepsilon{}^2)$ that are used to describe actual demand as in Eq. (A1).

$$\begin{aligned}
\hat{y}_{t+1} &= a_t + b_t(t+1) \\
&= \bar{y} + b_t(t+1-\bar{t}) \\
&= \frac{\sum_{i=0}^{t} y_i}{t+1} + \frac{\sum_{i=0}^{t}(t_i-\bar{t})y_i}{\sum_{i=0}^{t}(t_i-\bar{t})^2}(t+1-\bar{t}) \\
&= \frac{\sum_{i=0}^{t} y_i}{t^*} + \frac{(t^*-\bar{t})}{\sum_{i=0}^{t}(t_i-\bar{t})^2}\sum_{i=0}^{t}(t_i-\bar{t})y_i \qquad (let\ t+1=t^*) \\
&= \sum_{i=0}^{t} y_i \left[\frac{1}{t^*} + \frac{(t^*-\bar{t})}{\sum_{k=0}^{t}(t_k-\bar{t})^2}(t_i-\bar{t})\right] \\
&= \sum_{i=0}^{t} y_i A_i^t \qquad \left(let\ A_i^t = \left[\frac{1}{t^*} + \frac{(t^*-\bar{t})}{\sum_{k=0}^{t}(t_k-\bar{t})^2}(t_i-\bar{t})\right]\right) \\
&= \sum_{i=1}^{t} A_i^t \sum_{j=0}^{i-1} \varepsilon_j \qquad \left(as\ y_t = \sum_{i=0}^{t-1} \varepsilon_i,\ y_0 = 0\right) \\
&= \sum_{i=0}^{t-1} \varepsilon_i \sum_{j=i+1}^{t} A_j^t \\
&= \sum_{i=0}^{t-1} \varepsilon_i B_i^t \qquad \left(let\ B_i^t = \sum_{j=i+1}^{t} A_j^t = \sum_{j=i+1}^{t}\left[\frac{1}{t^*} + \frac{(t^*-\bar{t})}{\sum_{k=0}^{t}(t_k-\bar{t})^2}(t_j-\bar{t})\right]\right)
\end{aligned}$$

Appendix E: Modeling Autocorrelation of Future Forecast Errors

We describe how the lag 1 autocorrelation of future forecast errors is modeled. We build a first order autoregressive (AR(1)) model of future forecast errors:

$$E_t = c + \phi E_{t-1} + u_t \tag{A9}$$

where both E_{t-1} and E_t are normal random variables with mean zero and increasing variance over time.

c and ϕ are functions of these means ($m_{E_{t-1}}$ and m_{E_t}) and variances ($\sigma_{E_{t-1}}{}^2$ and $\sigma_{E_t}{}^2$), and the lag 1 autocorrelation function, ρ_1:

$$\begin{aligned} c &= m_{E_t} - \phi m_{E_{t-1}} = 0 \\ \phi &= \frac{\rho_1 \sigma_{E_t}}{\sigma_{E_{t-1}}} \end{aligned} \tag{A10}$$

We assume ρ_1 is the average of the lag 1 autocorrelation functions of the historical forecast errors. u_t is a noise term which is normally distributed with mean zero and standard deviation *s*. We define *s* as:

$$s = (1 - \rho_1{}^2)^{1/2} \sigma_{E_t} \tag{A11}$$

Given the historical forecast error at t=n, we model the future forecast error at t=n+1 by applying Eq. (A9) and repeat the process.