Percolation, Cascades, Rumors in Random Networks

Dan Whitney

June 15, 2009

Acknowledgements: Prof Christophe Midler and Prof Daniel Krob of Ecole Polytechnique de Paris

Network Percolation Contexts

- Spread of diseases (Watts and others)
- Propagation of rumors (Newman, Watts, Calloway)
- Success of “blockbusting” (Schelling)
- Decision to join a riot (Granovetter)
- Adoption of innovations (Rogers, Valente)
- In each case, nodes are assumed to be different in their susceptibility

Incumbent Percolation Theory

- Assumes an infinite network that is tree-like (not very densely connected)
- A node flips if any neighbor flips
- Predicts percolation if the network is barely connected
- Percolation occurs if the probability of a giant connected cluster = 1
  – Giant cluster: a finite fraction of the infinite network
- This is an existence proof that does not "look inside the box"
  – No time history, no mechanism

Theory and Simulations Are Inconsistent

- All simulations are finite and cannot reproduce the theoretical assumptions
- A “large” finite network is not a small infinite network
- Regardless, researchers use infinite-sparse-network theory to analyze finite non-sparse networks
- A theory is needed that directly addresses finite non-sparse networks

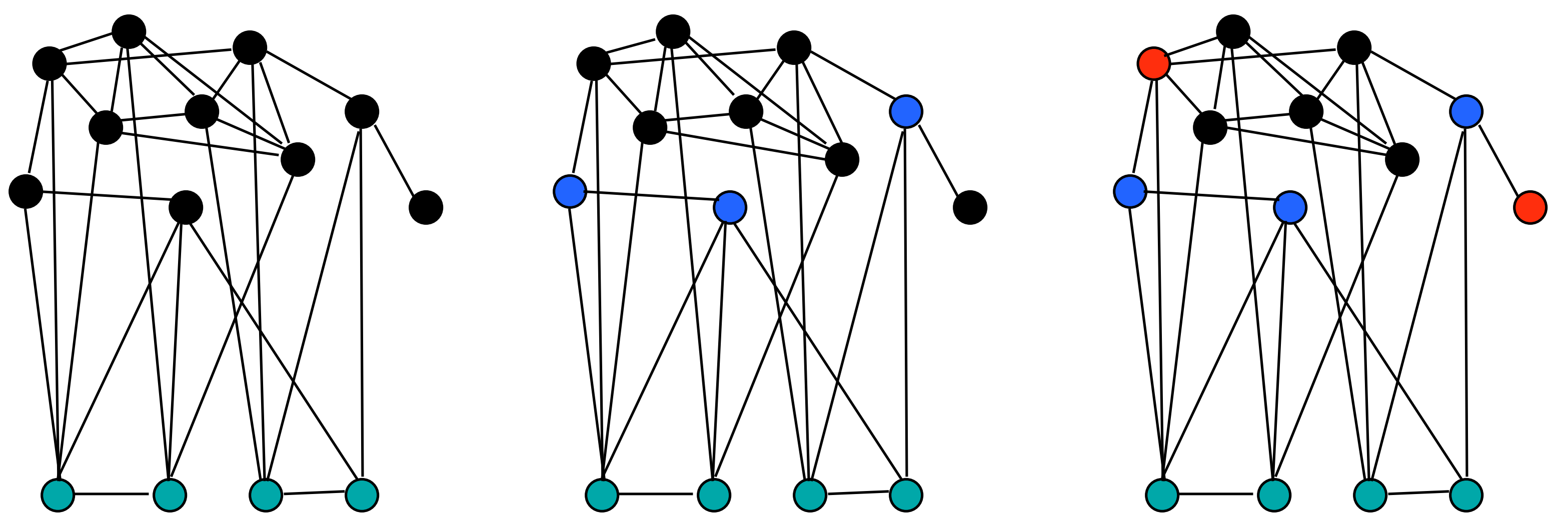
Percolation with a Threshold

- A seed node(s) is flipped from “off” to “on” arbitrarily
- Neighbors of flipped nodes will flip if a large enough fraction (exceeding a given threshold) of their neighbors flip
- The threshold divides the nodes into “k-classes”
  - Nodes are called *vulnerable* if one flipped neighbor will flip them - they have K* or fewer neighbors ($K^* = \lfloor 1/\phi \rfloor$ where ϕ = the threshold)
  - Nodes are called *first order stable* if it takes 2 flipped neighbors to flip them - they have between K*+1 and 2K* neighbors, etc.

Two Steps of Cascade with Threshold ϕ = 0.25, K* = 4

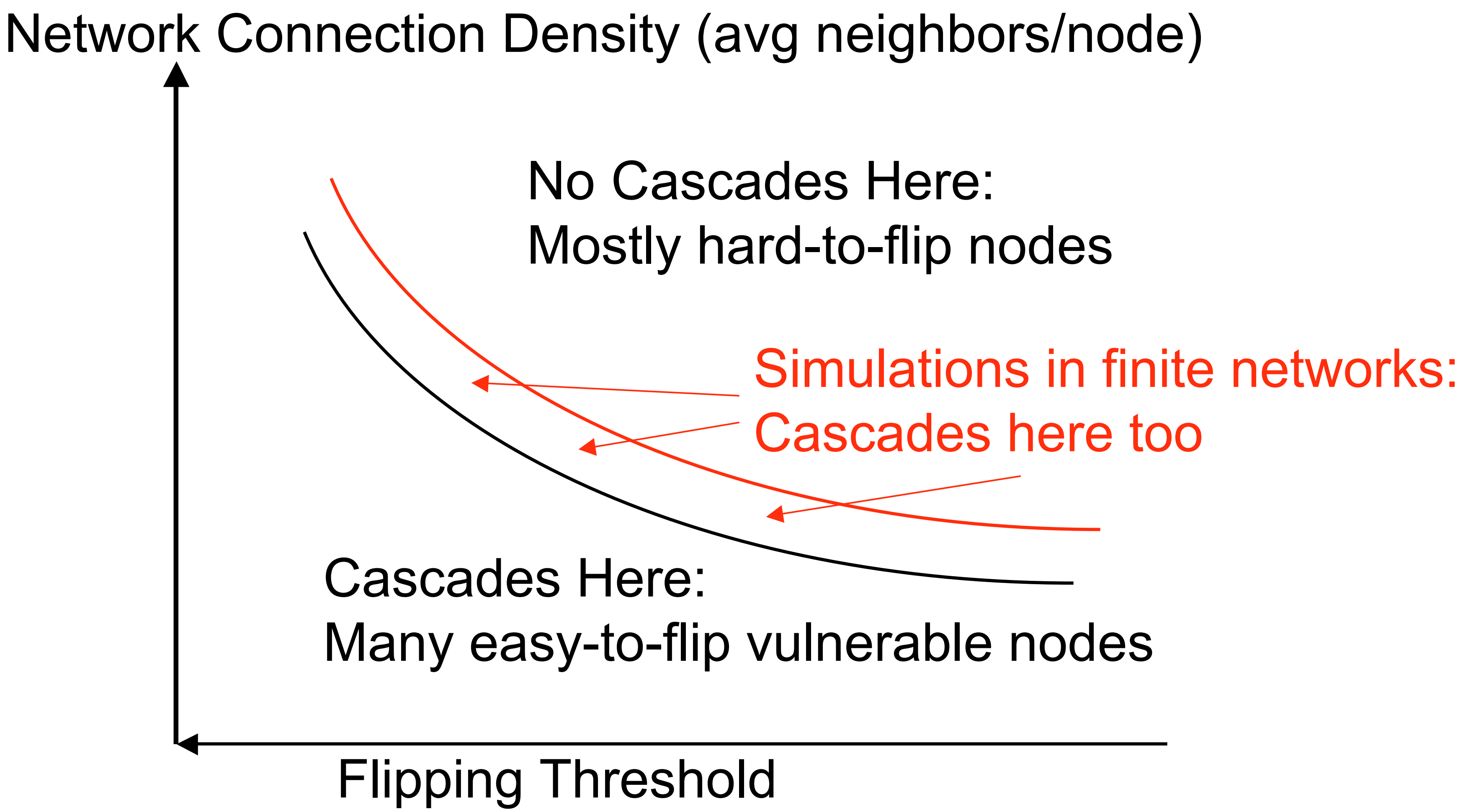
$1 \leq k \leq K^*$: flip if $\geq$ 1 neighbor flips

$K^*+1 \leq k \leq 2K^*$: flip if $\geq$ 2 neighbors flip



Seed First Step Second Step

Predictions of Existing Theory (Watts)



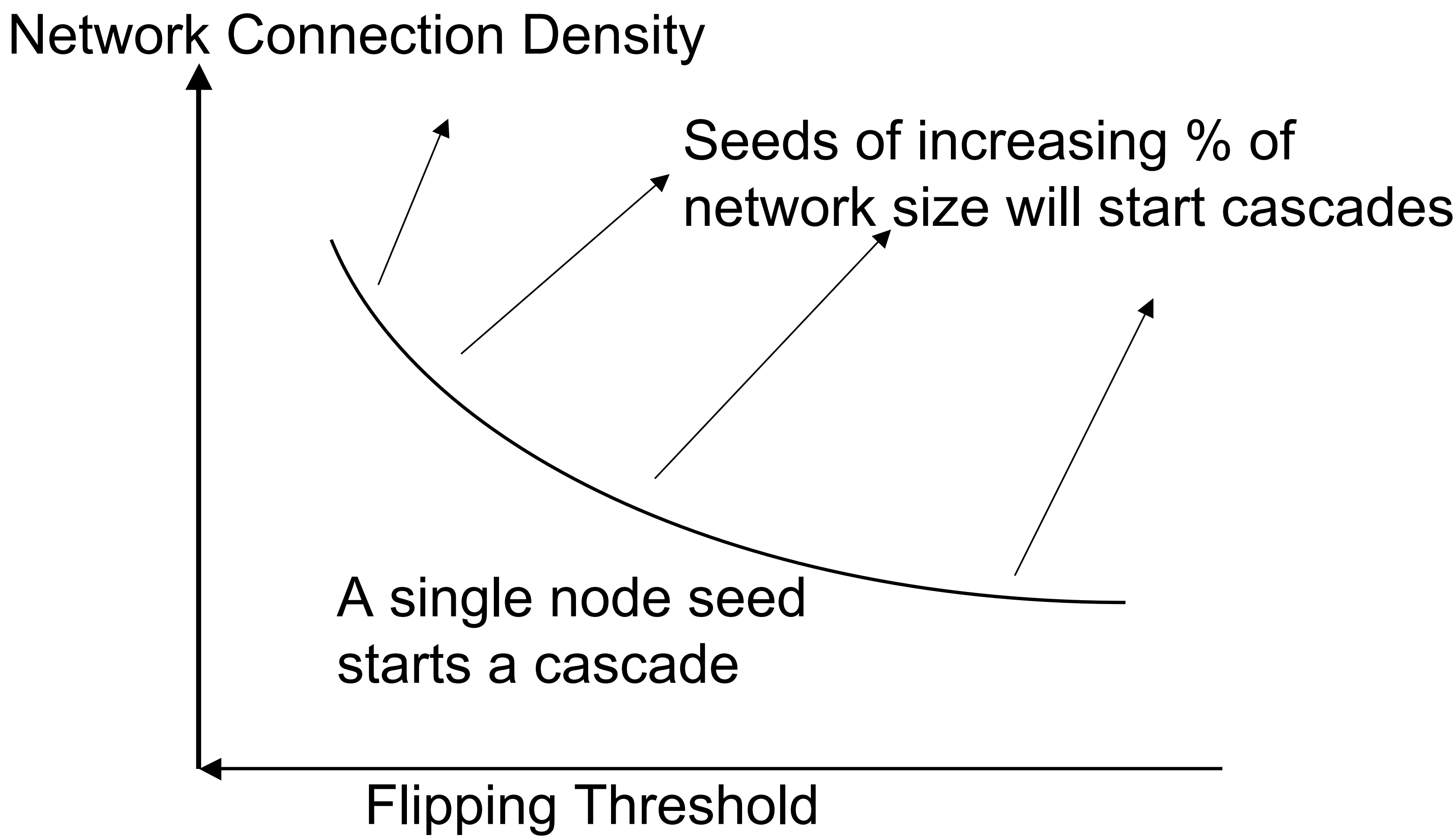
Goals of a New Theory

- Develop an analytical model of cascades in random networks with a threshold rule that deals directly with finite networks that are not tree-like
- Understand why cascades occur in the no-cascade region
- Look at seeds with more than one node and find out if big enough seeds can start cascades in the no cascades region
- Try to emulate the theory of adoption of innovations
  - Make the *k*-classes correspond to “early majority,” “late majority,” and “laggards”
  - The seed comprises the innovators

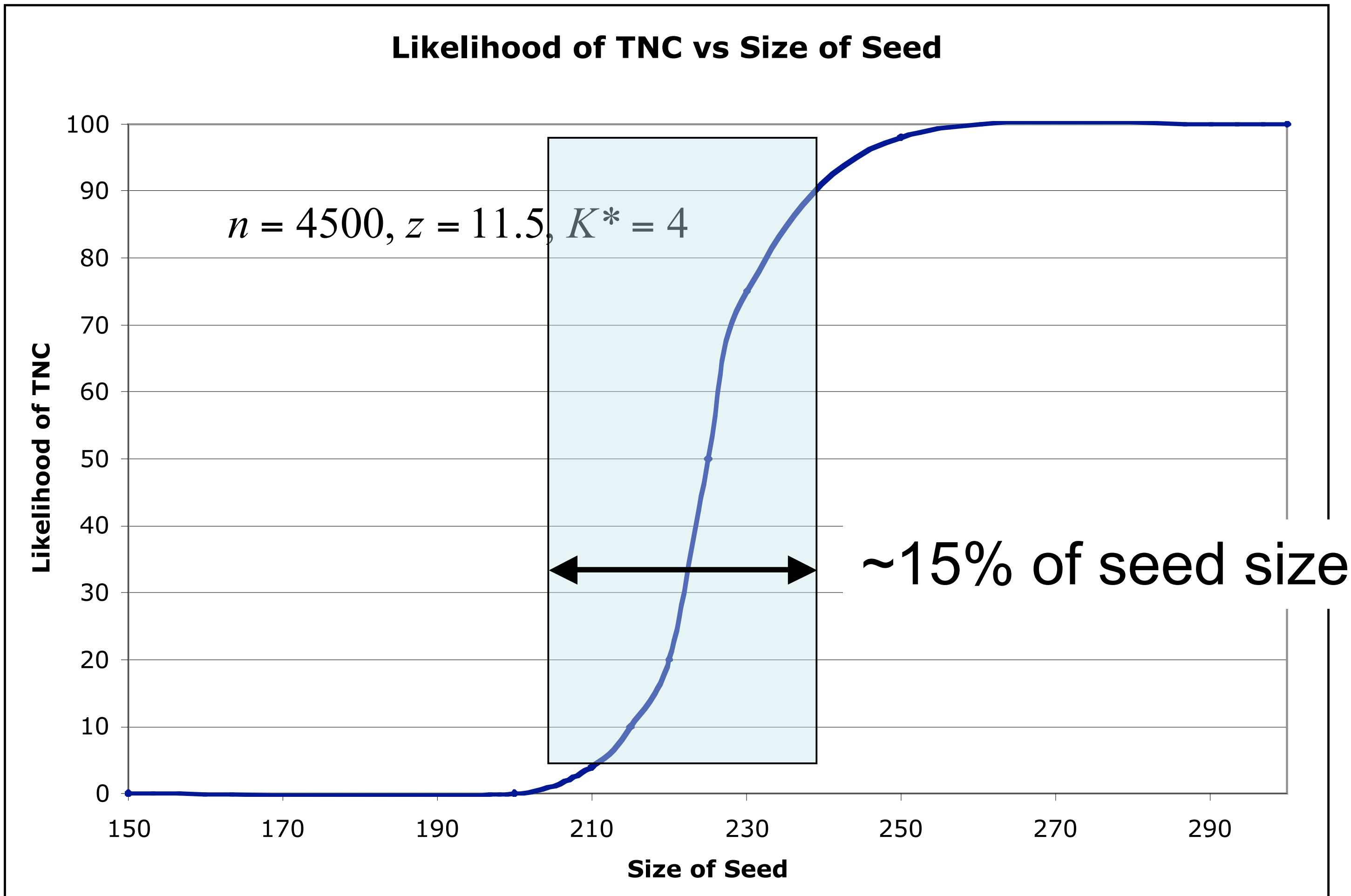
Simulation Results Reproduced by the New Theory

- There is no cutoff at the upper boundary
- Instead, larger size seeds can start cascades
- The transition from too small seed to big enough seed is sudden and constitutes a phase transition
- Seeds in the size transition range result in cascades that seem about to die but then take off explosively
  - A critical mass, a near death phenomenon

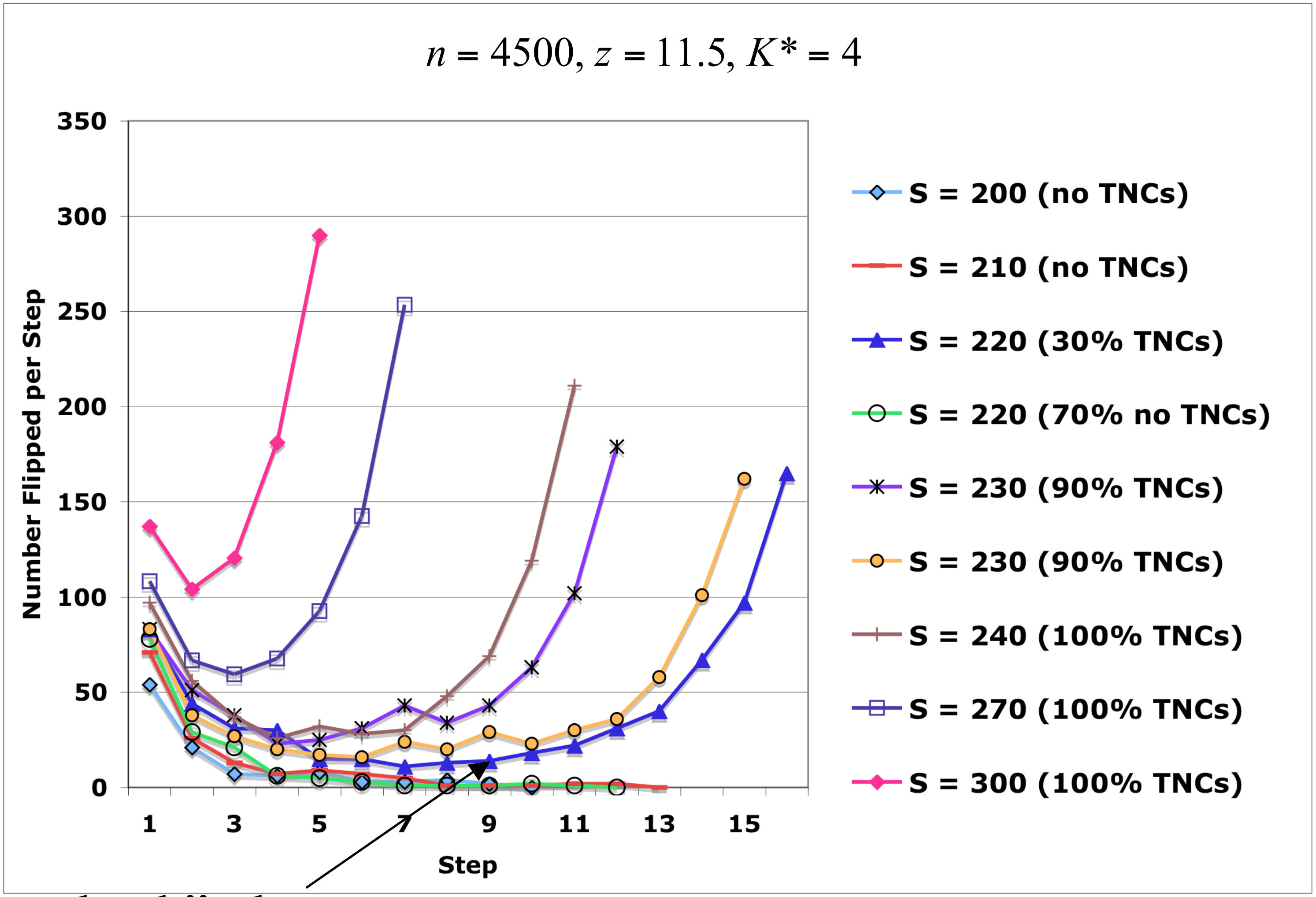
Result of Simulations and New Theory



Threshold Seed Size - A Phase Transition (Simulation)



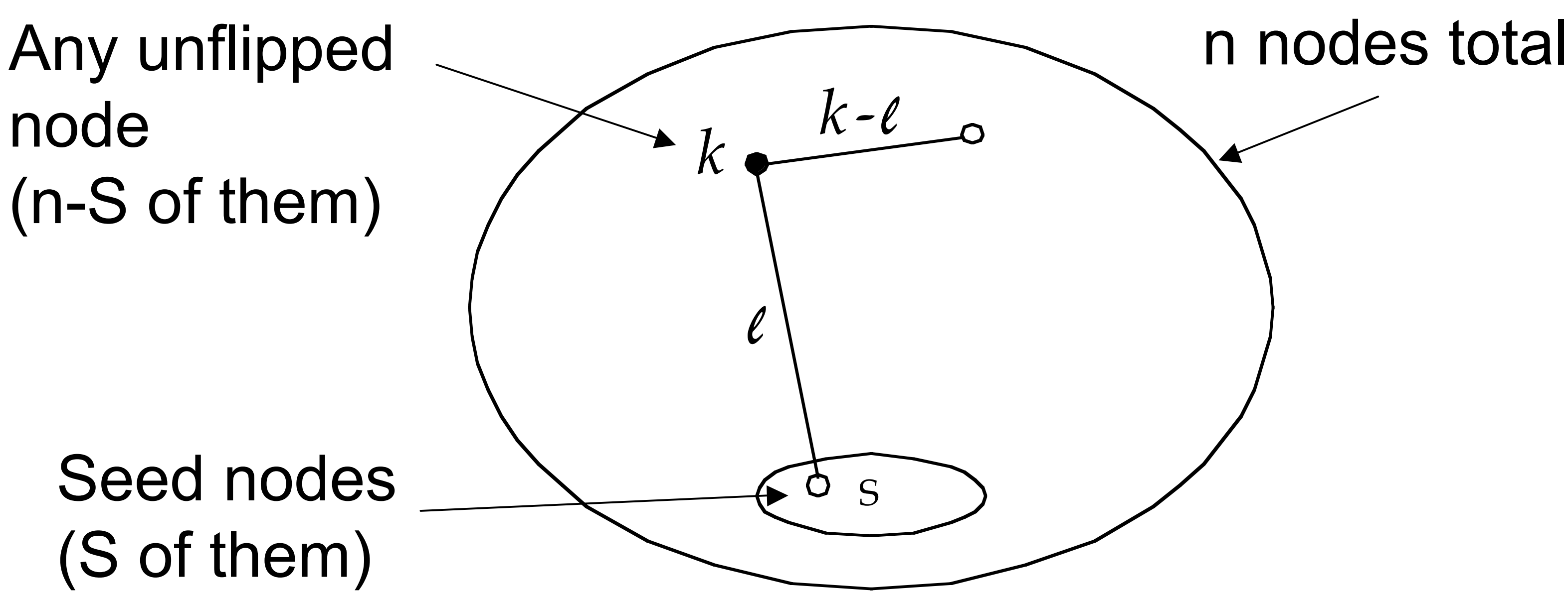
Typical Cascade Trajectories (Simulation)



"Near death" phenomenon

The Model (Initial Form) for Step 1

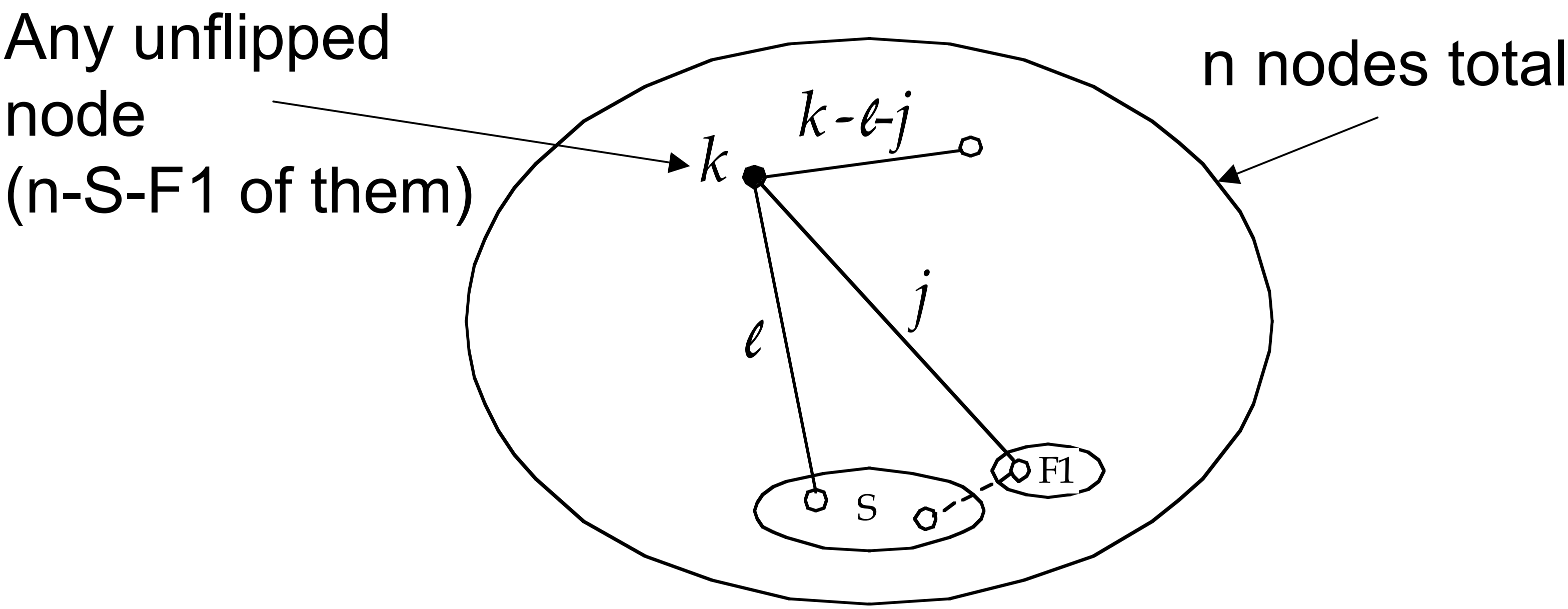
$$p_k = \sum_{\ell=0}^{k} \binom{S}{\ell} p^{\ell} (1-p)^{S-\ell} \binom{n-S-1}{k-\ell} p^{k-\ell} (1-p)^{n-S-1-(k-\ell)}$$



Nodes flipped by seed *S* are called *F1*

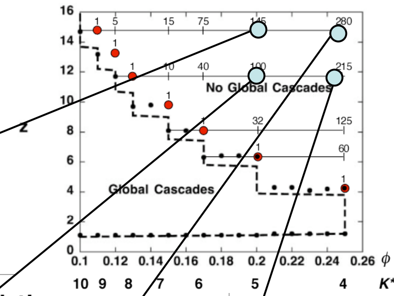
Model (Initial Form) for Step 2

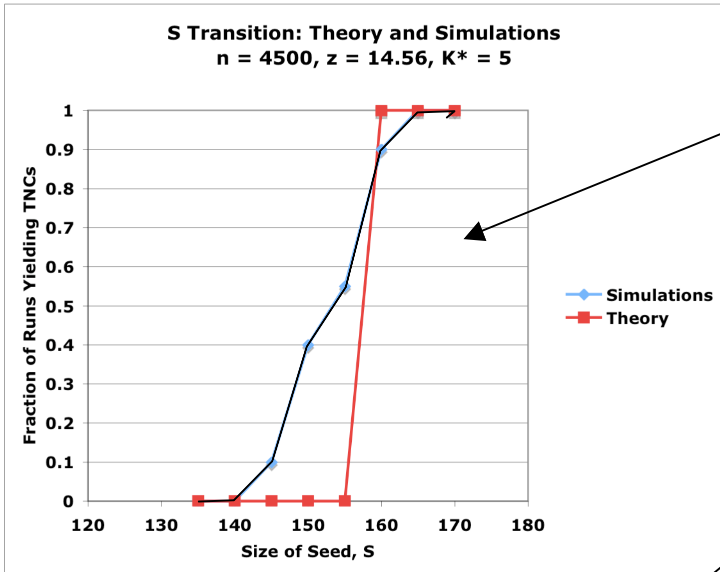
$$p_k = \sum_{\ell=0}^{k} \sum_{j=0}^{k-\ell} \binom{S}{\ell} p^{\ell} (1-p)^{S-\ell} \binom{F1}{j} p^{j} (1-p)^{F1-j} \binom{n-S-F1-1}{k-\ell-j} p^{k-\ell-j} (1-p)^{n-S-F1-1-(k-\ell-j)}$$

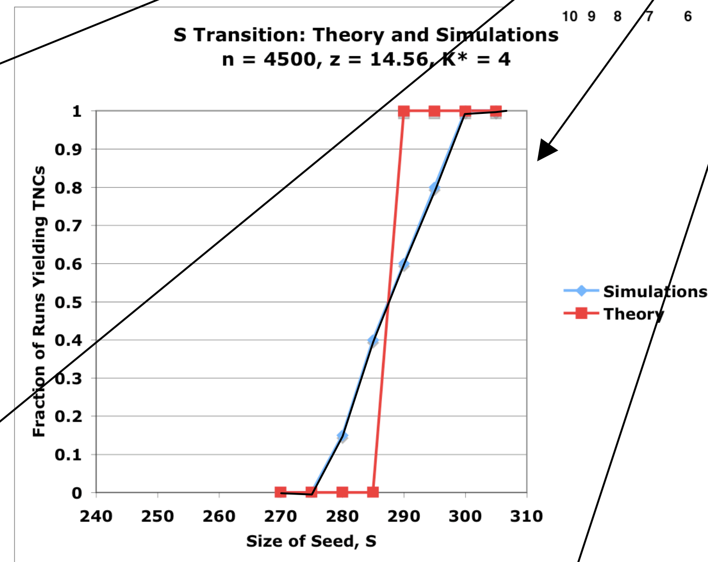


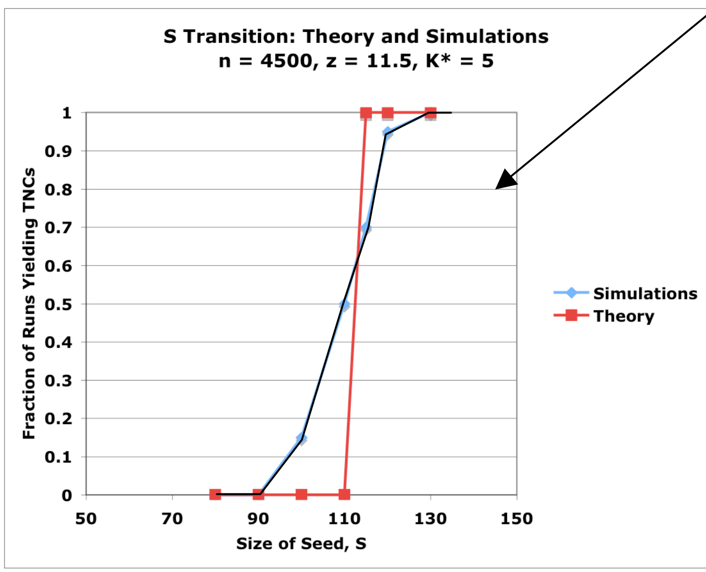
Nodes flipped by *F1* and unused edges from *S* are called *F2*

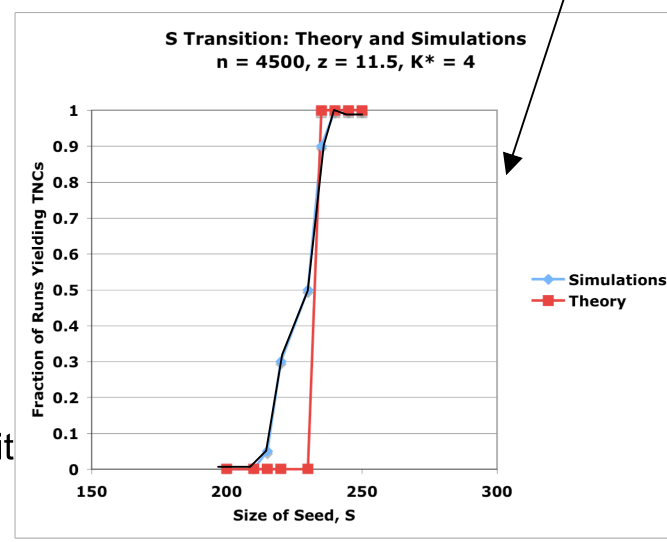
Ability to Predict Threshold Seed Size



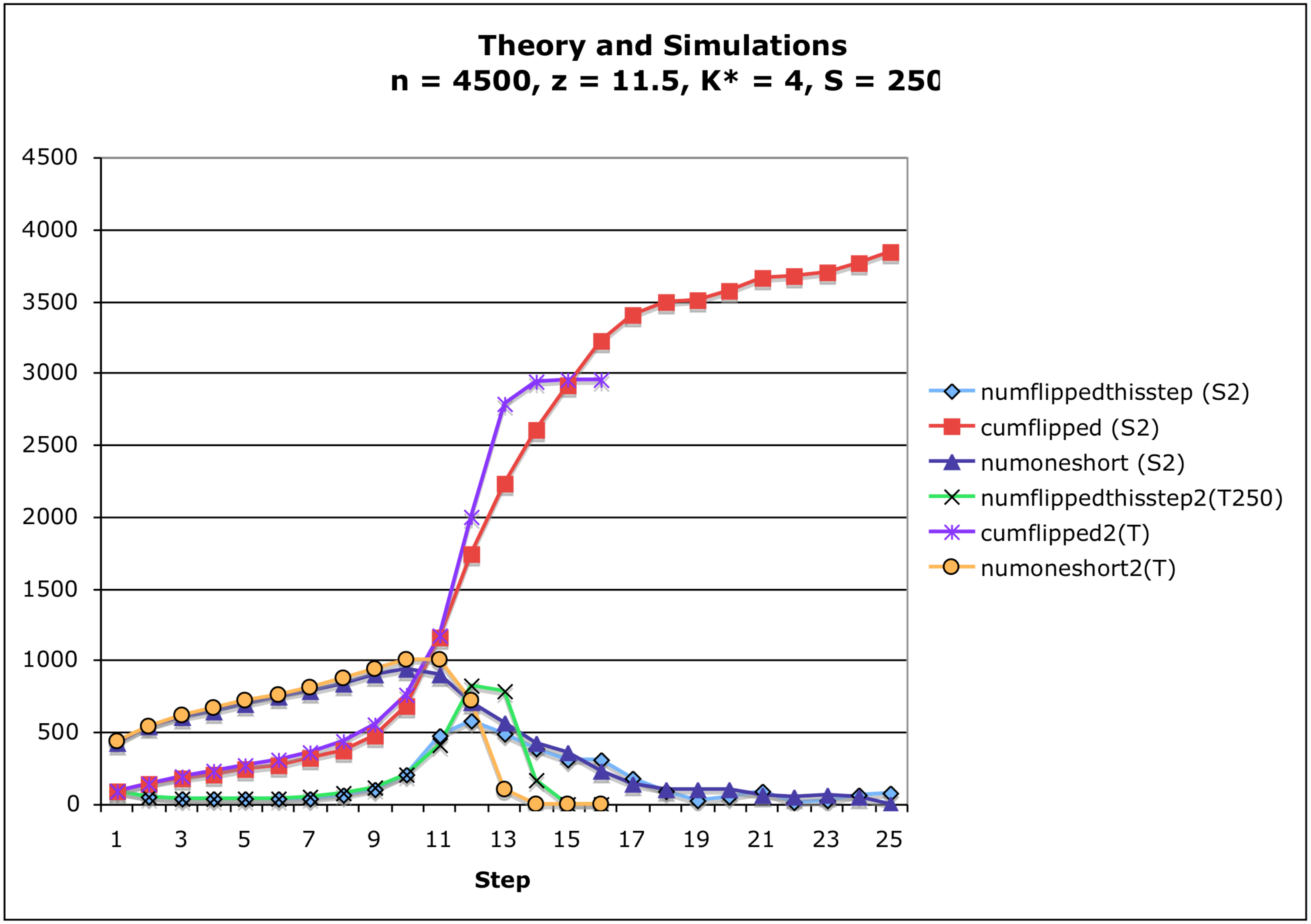






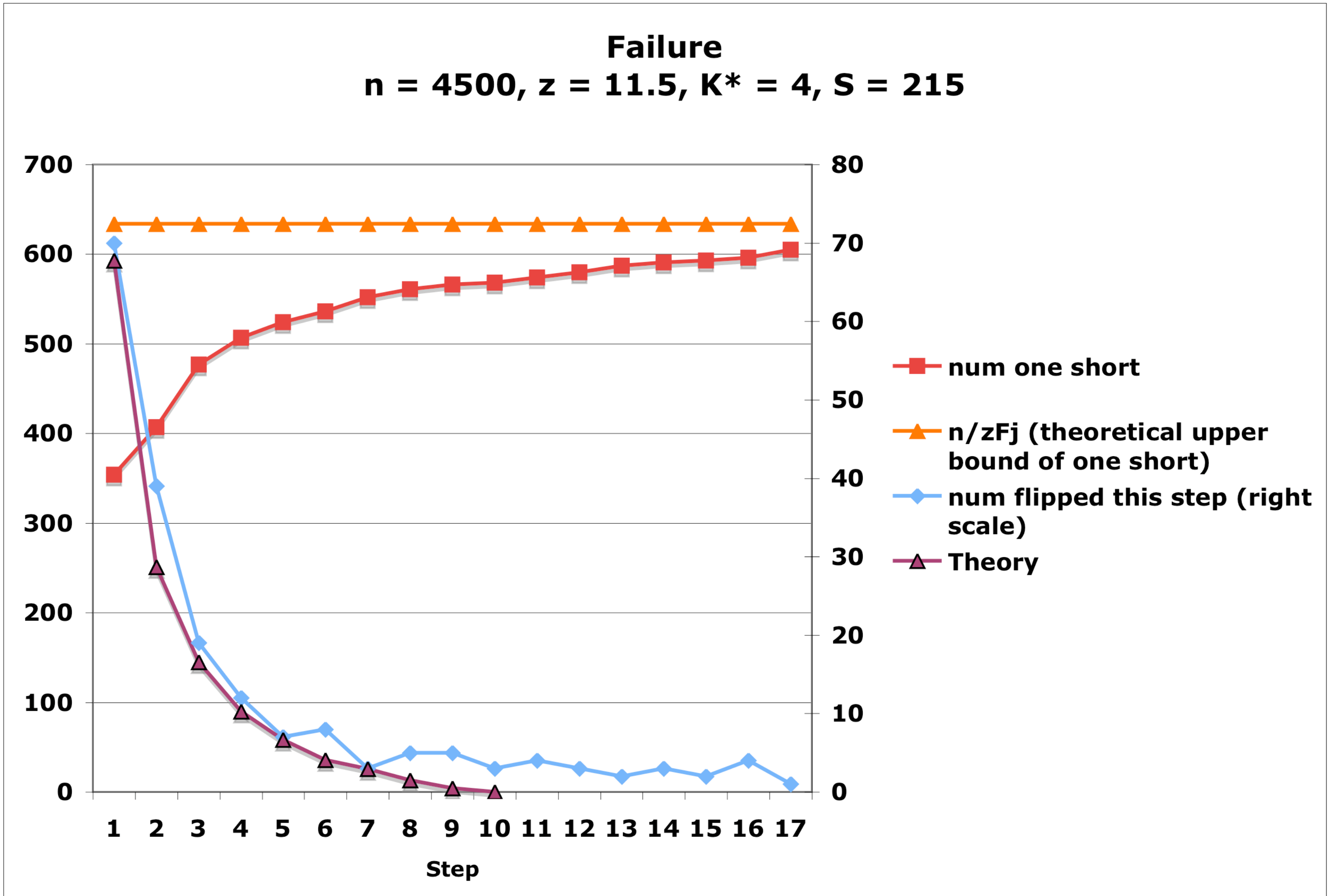


Full Evolution: Theory and Simulation

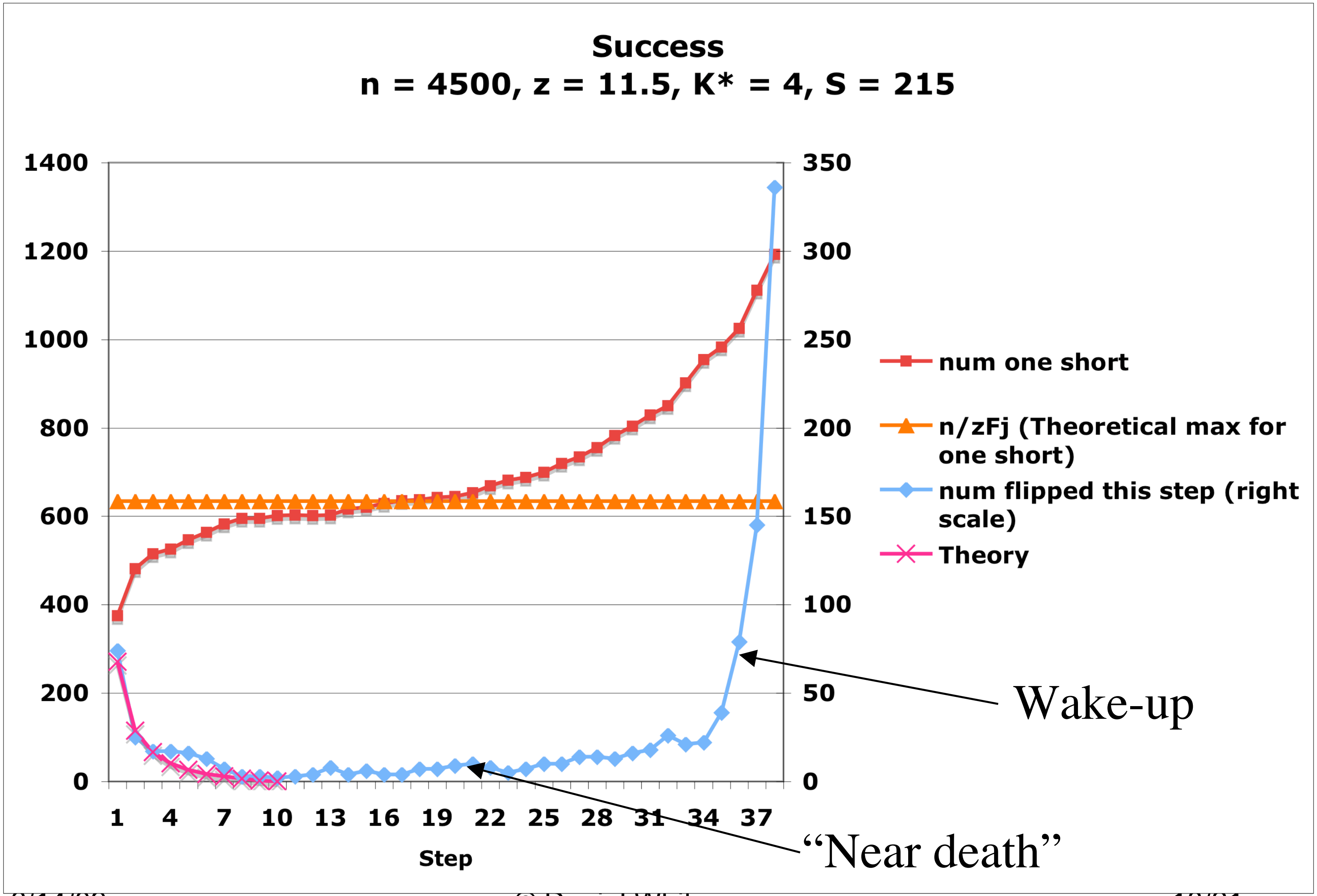


Theory agrees well with simulation but runs into numerical difficulties after step 12.

At S = 215, Failure Most of the Time



Occasionally, Success: Why?



Cause of Wake-up After Near Death

- Caused by a critical mass phenomenon
- Near death, only a few nodes flip on each step
- At most they can hit one node each since, for so few nodes, the likelihood of multiple hits is about zero
- So only nodes that are one short of flipping have any chance to flip during this phase
- This chance is proportional to how many one-shorts there are on any step
- This population is growing but at the same time the number of flippers is falling

$$min\ one\_short = \frac{n}{avg\ edges\ of\ \ flipped\ nodes}$$

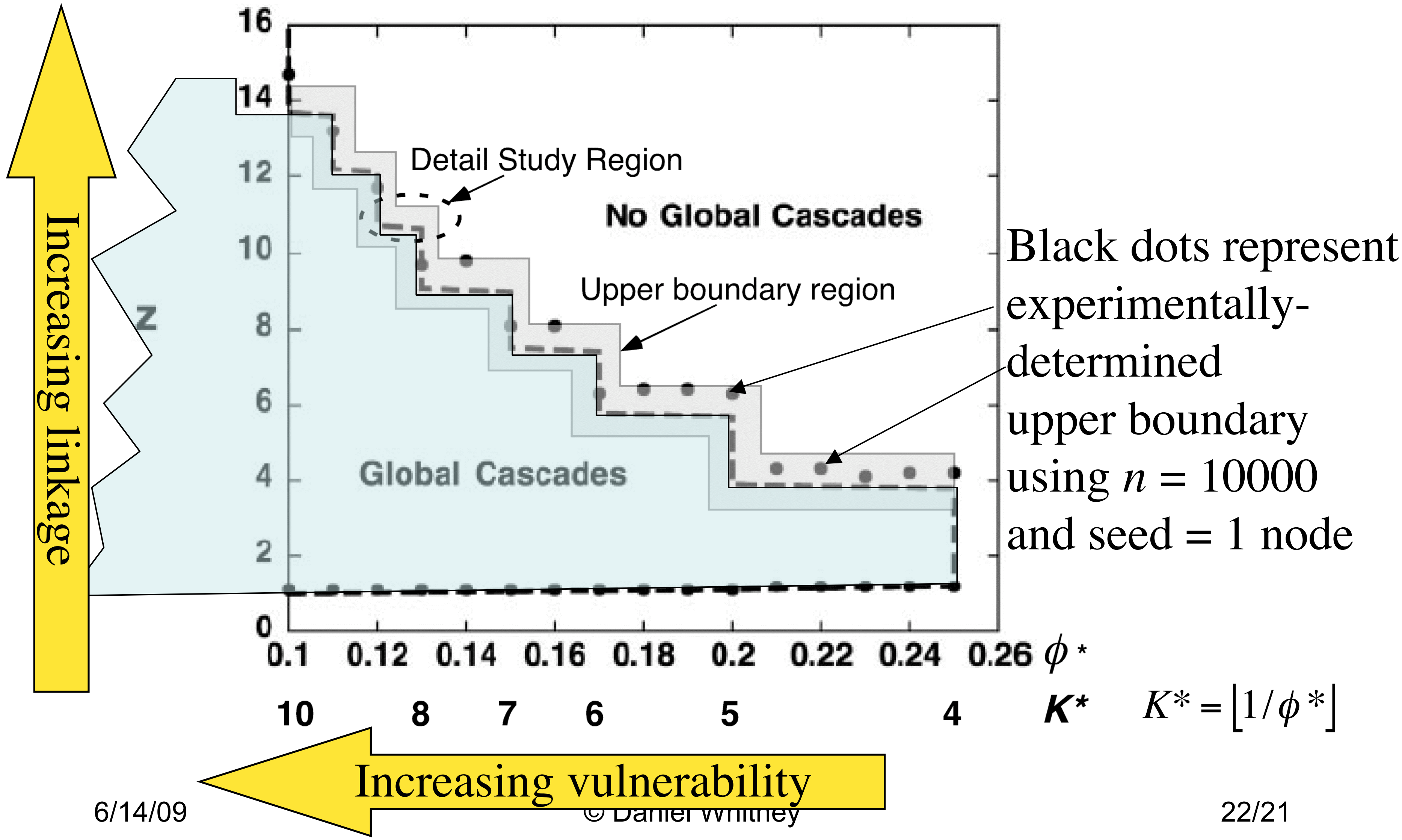
Conclusions

- The new dynamic model of cascades in finite dense random networks with thresholds is reasonably accurate
- It reproduces the typical S-curve of other theories
- It makes only one incorrect assumption vs many for the incumbent theory
- It is a Markov model, a construction proof
  - You can see inside the box
- It reproduces the main features of simulations
  - Critical seed size as a function of *n*, *z* and *K**
  - Near death behavior
- It is cumulative and subject to numerical drift

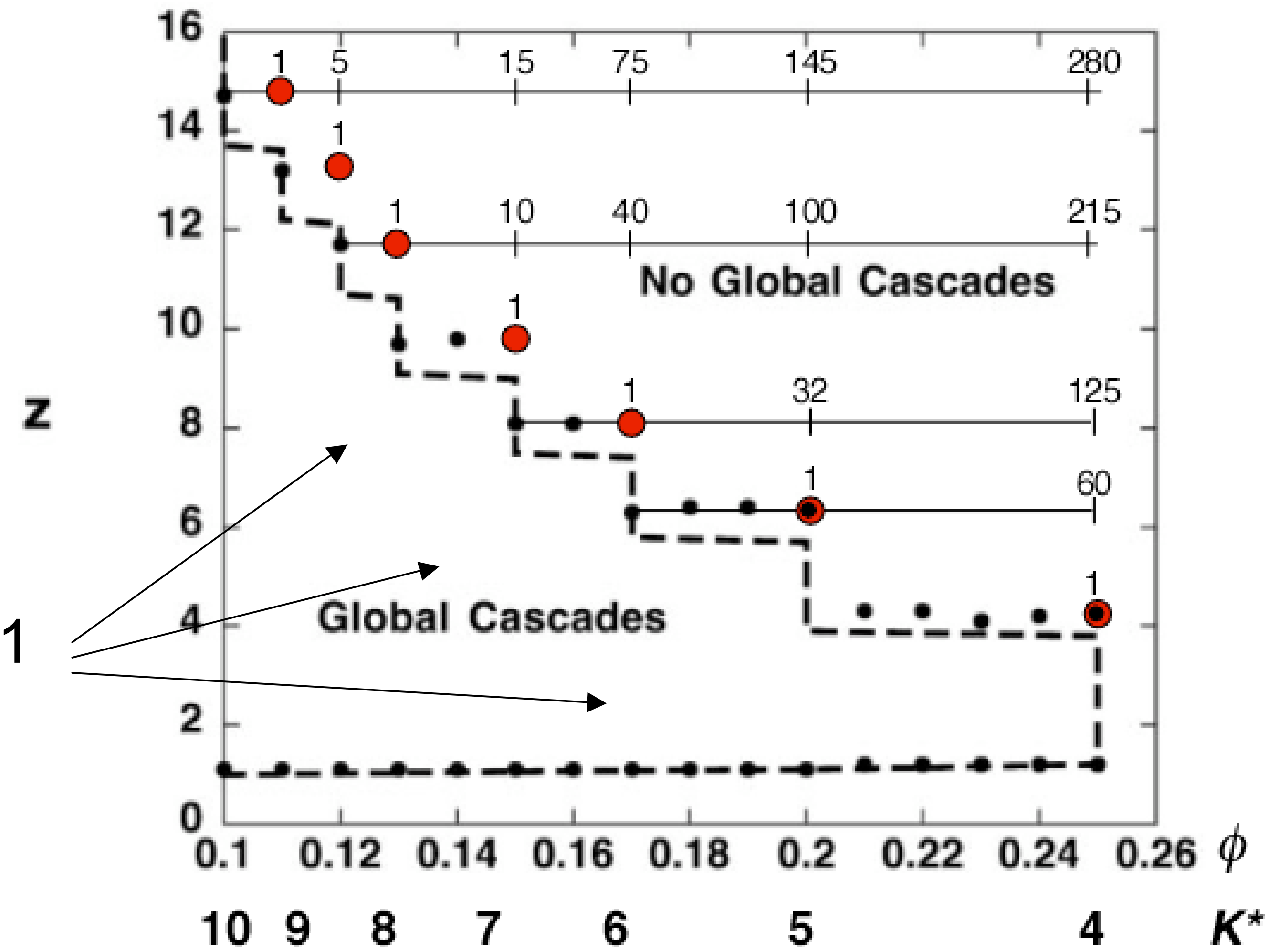
Observations

- Innovation diffusion theory assumes that an innovation will take off if there is a critical mass of innovators
- My model says that for seeds in the transition region, the critical mass is not innovators but potential adopters of all degree classes who are just short of being convinced
- Variation plays a huge role in the behavior of the simulated networks, indicating that there are differences between them even though theory says that they are described by the mean only

Watts' Cascade Window for Random Networks



Threshold Seed Sizes in No Cascades Region (Simulation with 4500 nodes)

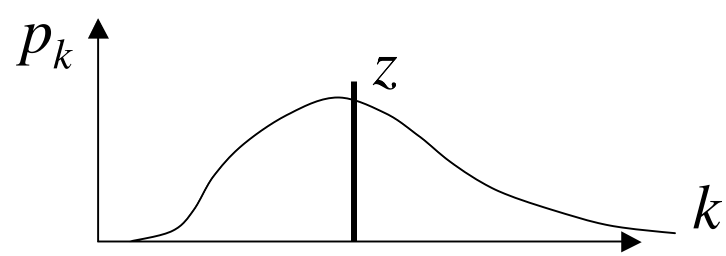


Seed = 1 here

Random Networks

- Network has *n* nodes
- Pick pairs of nodes and connect them with probability *p*
- The result is a network with a binomial degree distribution (p_k= probability that a node has k neighbors)

$$p_k = \binom{n-1}{k} p^k \left(1-p\right)^{n-1-k}$$



Progression of a Cascade: Each Flipped Node Flips its Immediate Neighbors