Inter-airline Equity and Airline Collaboration in Airport Scheduling Interventions

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Abstract

In the absence of opportunities for capacity increases or operational enhancements, air traffic congestion can be significantly mitigated only through scheduling interventions aimed to control the imbalances between demand and capacity at busy airports. In this paper, we design, optimize, and assess a novel non-monetary mechanism for airport scheduling interventions that ensures inter-airline equity and enables airline collaboration. Under the proposed mechanism, the airlines would provide their preferred schedules of flights, their network connections and the relative scheduling flexibility of their flights to a central decision-maker, who may then suggest some scheduling adjustments to reduce anticipated delays. We develop a lexicographic modeling architecture that optimizes such interventions based on efficiency (i.e., meeting airline scheduling preferences), equity (i.e., balancing scheduling adjustments fairly among the airlines), and on-time performance (i.e., mitigating airport congestion) objectives, subject to scheduling and network connectivity constraints. Theoretical and computational results suggest that (i) inter-airline equity can be achieved at no, or small, losses in efficiency, and (ii) accounting for relative preferences of the airlines regarding the schedule modifications of different flights can significantly improve the outcome of scheduling interventions.

\textit{Keywords:} airport demand management, inter-airline equity, integer programming, dynamic programming, queuing model, mechanism design

1. Introduction

The development of air transportation systems worldwide has been supported by airport and air traffic management infrastructure. However, limitations on infrastructure capacity, coupled with significant growth in air traffic, have resulted in severe congestion at many of the world’s busiest airports. This congestion typically materializes in the form of flight delays and cancellations. The costs of air traffic congestion in the United States were estimated at over $30 billion for the year
2007 (Ball et al., 2010) and this issue is likely to become even more pressing over the medium- and long-term horizons as demand for air traffic is expected to increase nationally and internationally. Most of air traffic delays in the United States originate from imbalances between demand and capacity at the busiest airports, resulting from airlines scheduling more flights than available capacity. This paper proposes and assesses flight scheduling mechanisms for allocating airport capacity to the airlines to limit the extent of overscheduling at peak hours, while minimizing interference with airlines’ competitive scheduling practices. Before presenting the contributions of this paper (Section 1.3), we elaborate on the approaches for airport congestion mitigation (Section 1.1) and review the available mechanisms for resource allocation (Section 1.2).

1.1. Airport Congestion Mitigation

Mitigating flight delays requires better matching of demand and capacity. Unfortunately, airport capacity cannot be substantially scaled up in the short- and medium-term, given infrastructural and operational constraints faced in the dense urban areas and airspaces. Imbalances between demand and capacity can thus only be managed through (i) improvements in the utilization of airport capacity, or (ii) scheduling interventions (Barnhart et al., 2012b). Capacity utilization refers to interventions aimed to adjust dynamically the flow of aircraft over each day of operations. Typical procedures include the sequencing and spacing of aircraft (Balakrishnan and Chandran, 2010; Solveling et al., 2011), the selection of active runways and the balancing of the arrival and departure loads at an airport (Gilbo, 1993; Bertsimas et al., 2011b; Jacquillat et al., accepted for publication, 2015), the ground holding of aircraft (Odoni, 1987; Vranas et al., 1994; Ball et al., 2003), and the airborne holding, rerouting and speed control of en-route aircraft (Bertsimas and Stock Patterson, 1998; Bertsimas et al., 2011c). These interventions can reduce the magnitude and/or the costs of air traffic delays, but are generally insufficient when demand exceeds capacity by any significant margin. Scheduling interventions, in contrast, refer to the demand management measures that limit, or constrain, the number of flights scheduled at an airport. They are typically implemented months in advance of the day of operations (before flight schedules get published and tickets get marketed). This is the main focus of this paper.

Worldwide demand management practices differ widely. Most airports outside the United States operate under slot control policies that limit the number of flights scheduled per hour (or other units of time) and distribute a corresponding number of slots across the different airlines through an administrative procedure (International Air Transport Association, 2013). In contrast, no demand management is applied at a large majority of US airports. A few airports were subject to slot restrictions under the High Density Rule, until its phase-out in 2007. Since then, airline schedules of flights in the United States have been subject to limited constraints. As a result of these regulatory differences, US airports achieve a larger throughput than their European counterparts, but they also face larger imbalances between demand and capacity and hence, larger delays (Morisset and Odoni, 2011). “Flight caps” have been recently imposed at the three major airports in the New
York Metroplex, but these have been only loosely enforced and were found too high to effectively alleviate congestion (Office of Inspector General, 2010; Government Accountability Office, 2012; de Neufville and Odoni, 2013).

Recent research has showed the potential of scheduling interventions to mitigate congestion and improve performance at US airports. Using a game-theoretic framework of airline frequency competition, Vaze and Barnhart (2012b) showed that small reductions in allocated airport capacity can reduce delays, improve airline profitability, and increase passenger welfare. By modeling the trade-off between flight delays and passenger schedule delay (i.e., schedule inconvenience), Swaroop et al. (2012) also found that a reduction in allocated capacity of 10% to 20% would improve passenger welfare at a majority of busy US airports. Using a different approach, Pyrgiotis and Odoni (Articles in advance, 2015) and Jacquillat and Odoni (accepted for publication, 2015) modeled and optimized the effects of scheduling interventions on airline timetabling of flights. Starting from an unconstrained schedule, they generated smoother schedules that reduce peak-hour scheduling levels, and showed that mild scheduling changes could yield significant delay reductions. In summary, evidence suggests that performance improvements could be achieved through limited scheduling interventions at the busiest US airports.

However, the implementation of such scheduling interventions has remained limited in the United States. Two likely reasons for this are that many of the proposed scheduling adjustments do not take explicitly into account airlines’ scheduling preferences and priorities, and may be perceived to be penalizing one airline (or a small number of airlines) disproportionately. This paper aims to address these two concerns by explicitly integrating airline scheduling preferences and inter-airline equity considerations into the design of mechanisms for airport scheduling interventions.

1.2. Distributed Decision-making and Equity in Resource Allocation

Airport scheduling interventions fall into a broader class of problems involving the allocation of scarce resources by a central decision-maker to decentralized agents (here, airlines). Two major challenges in these problems are: (i) designing mechanisms through which agents can provide their utilities and preferences, and (ii) defining the objective of resource allocation (Sen et al., 2014).

First, each airline’s preferences on flight timetabling rely on private information about its network of flights, the composition of its fleet, the availability of its crew, its anticipated passenger demand, its business model, etc. A common criticism of the administrative slot control policies in place outside the United States is that they do not enable the airlines to provide their scheduling preferences and, in turn, may not result in the highest-valued flights being scheduled at the highest-valued times. To address this problem, market mechanisms based on congestion pricing (Carlin and Park, 1970; Daniel, 1995; Brueckner, 2002; Vaze and Barnhart, 2012a) or slot auctions (Rassenti et al., 1982; Ball et al., 2006; Harsha, 2009) have been proposed, but they have not been successfully implemented in the current institutional environment. To the best of our knowledge, airline
preferences regarding the relative scheduling flexibility of their flights have not been incorporated into non-monetary airport scheduling mechanisms.

Second, defining the central decision-maker’s objective in resource allocation may involve trade-offs between efficiency (i.e., maximizing the sum of agents’ utilities), equity (i.e., balancing utilities fairly among the agents), and, possibly, other objectives (e.g., maximizing outcome predictability, ensuring incentive-compatibility, etc.). The trade-off between efficiency and equity has been recently formalized by Bertsimas et al. (2011a, 2012), who obtained with general utility functions theoretical bounds on the “price of fairness” and the “price of efficiency”, i.e., the relative loss in efficiency if equity is maximized, and conversely. However, no study till date has incorporated inter-airline equity considerations into the design of airport scheduling mechanisms.

In a related area, non-monetary, distributed, and equitable mechanisms have been developed for allocating air transportation capacity through Air Traffic Flow Management (ATFM) initiatives. ATFM consists of optimizing the flows of aircraft at airports or through air traffic control sectors over the day of operations to reduce local imbalances between demand and capacity. Whereas early ATFM developments were centralized, and based on efficiency objectives exclusively (minimizing total congestion costs), their successful implementation was made possible by involving the airlines in the decision-making process. First, the Collaborative Decision Making (CDM) paradigm decentralized operating decisions to the airlines whenever possible, through airline collaboration and slot trading, to determine which flights to delay and which flights to prioritize (Vossen and Ball, 2006). Second, recent studies incorporated inter-airline equity considerations into the objective of ATFM models, aiming to make the outcome of centralized decision-making more acceptable to each of the individual airlines (Barnhart et al., 2012a; Bertsimas and Gupta, 2015). This paper integrates these two objectives into the design and optimization of scheduling interventions (months in advance). A stream of more recent studies have proposed mechanisms to gather airlines’ operating preferences and to integrate them into the broader decisions related to the duration, magnitude and scope of the ATFM initiatives (Swaroop, 2013; Ball et al., 2014; Evans et al., Articles in advance, 2015). Similar mechanisms could be applicable for integrating airline preferences into the broader decisions related to the design of scheduling interventions (e.g., the total number of flights to reschedule and the corresponding delay reductions to expect), but they are not considered in this paper and are also left for future research.

The scheduling interventions studied in this paper exhibit several differences from the ATFM problem. First, the information available at the time of flight scheduling is more limited than that available on the day of operations. Second, unlike in ATFM, no standard of equity has been accepted in the industry with respect to scheduling interventions. Third, scheduling interventions may result in flights being rescheduled later or earlier than their preferred scheduled times requested by the airlines. This contrasts with the situation in ATFM where flights cannot be moved earlier than their scheduled time. Thus, the ATFM schemes of ration-by-schedule and schedule compression do not
have any direct analogs in the context of scheduling interventions. It is thus necessary to propose new metrics of inter-airline equity, to design new mechanisms enabling the airlines to provide their scheduling preferences, and to develop new modeling frameworks for scheduling interventions.

An important challenge in mechanism design is to ensure that agents have incentives to provide their inputs truthfully. However, we do not explicitly consider in this paper the game-theoretic aspects of the submission of airline scheduling preferences into the proposed mechanisms. Given the difficulties associated with quantifying the differences in airline operating profitability under only marginally different schedules of flights, we leave this question for future research. Instead, in this paper, we identify and discuss those aspects of our mechanisms which could potentially create incentives for airlines to deviate from submission of their true preferences, and strive to minimize these opportunities by minimizing interference with airline competitive scheduling.

1.3. Contributions

The main contribution of this paper consists of formulating and implementing a set of optimization models for the design of mechanisms for airport scheduling interventions. These mechanisms involve the airline community providing scheduling inputs to a central decision-maker (e.g., administratively appointed schedule coordinators at slot-controlled airports, the Federal Aviation Administration (FAA) in the United States), who then produces a schedule of flights to reduce anticipated delays at a considered airport. Our approach builds upon the Integrated Capacity Utilization and Scheduling Model (ICUSM) developed by Jacquillat and Odoni (accepted for publication, 2015) that optimizes such schedule modifications, but extends it in a way that accounts for inter-airline equity and airline collaboration. The resulting mechanisms (i) rely on non-monetary transfers exclusively; (ii) enable the airlines to provide their scheduling preferences; and (iii) balance scheduling adjustments equitably among the airlines.

Specifically, this paper makes the following four contributions:

- **Proposing a set of performance attributes for scheduling interventions.** We identify efficiency (i.e., meeting airline scheduling preferences), equity (i.e., balancing scheduling adjustments fairly among the airlines), and on-time performance (i.e., mitigating airport congestion) as three performance attributes. We develop quantitative indicators for each of them, using a unified framework of scheduling interventions. We then formulate a tractable lexicographic architecture to characterize the trade space between efficiency, equity, and on-time performance in airport scheduling interventions.

- **Discussing the trade-off between efficiency and equity.** First, we show that, under some conditions on the scheduling inputs provided by the airlines, efficiency and equity can be jointly maximized. We then describe how a trade-off between efficiency and equity may arise from inter-airline variations in flight schedules, network connectivities and flight valuations.
• Proposing non-monetary, credit-based mechanisms for airport scheduling interventions. These mechanisms enable the airlines to provide to the central decision-maker their preferred schedule of flights and the relative scheduling flexibility of their flights through credit allocation. These inputs are then used to generate a modified schedule, based on efficiency, equity, and on-time performance objectives.

• Generating computational scenarios and results at the John F. Kennedy Airport (JFK) to analyze the performance of the proposed mechanisms. We show that, under current scheduling conditions, inter-airline equity can be achieved in airport scheduling interventions at no (or minimal) efficiency losses. We then show that accounting for airlines’ scheduling preferences can significantly improve the outcome of scheduling interventions.

1.4. Outline

The remainder of this paper is organized as follows. In Section 2, we summarize the Integrated Capacity Utilization and Scheduling Model (ICUSM), and discuss its main limitations. In Section 3, we formulate a model of airport scheduling interventions that builds upon the ICUSM to account for inter-airline equity and airline scheduling preferences. Section 4 discusses the conditions under which efficiency and inter-airline equity can be jointly maximized. In Section 5, we propose non-monetary, credit-based mechanisms that define the inputs provided by the airlines and the subsequent scheduling interventions. In Section 6, we apply the models and mechanisms developed in this paper to a case study at JFK Airport. Section 7 concludes.

2. Base Model of Scheduling Interventions

We first summarize the Integrated Capacity Utilization and Scheduling Model (ICUSM) from Jacquillat and Odoni (accepted for publication, 2015), which provides the methodological basis for the design and optimization of the scheduling mechanisms in this paper. Moreover, this section introduces several notations that will be used throughout this paper.

2.1. Formulation

The ICUSM considers a simple two-step scheduling process, under which the airlines provide a schedule of flights to a central decision-maker, who may then propose scheduling adjustments to reduce above-capacity scheduling at an airport, and hence anticipated delays. We denote by II the airport where the scheduling interventions are considered. Consistent with practice at US airports, no flight is eliminated, and delays are reduced by distributing flights more evenly over the day. The model takes as inputs each airline’s preferred schedule of flights (e.g., the schedule in the absence of demand management) and estimates of the capacity of airport II (i.e., the expected number of movements that can be operated per unit of time in various operating conditions). It determines which flights to reschedule to later or earlier times to minimize the displacement from airlines’
preferred schedule of flights, subject to scheduling and network connectivity constraints, which ensure that the airlines’ flight networks are minimally affected, and on-time performance constraints, which ensure that expected arrival and departure queue lengths do not exceed prespecified targets.

The ICUSM relies on a modeling framework that integrates into an Integer Programming model of scheduling interventions a Stochastic Queuing Model of airport congestion and a Dynamic Programming model of airport capacity utilization. We first describe the Integer Programming scheduling framework, and then the models of airport operations used to formulate the on-time performance constraints.

**Inputs.**

\[ T = \{ t \} \text{ set of 15-minute time periods, indexed by } t = 1, \ldots, T \]
\[ F = \{ i \} \text{ set of flights, indexed by } i = 1, \ldots, F \]
\[ F_{\text{arr}} / F_{\text{dep}} = \{ i \in F \} \text{ set of flights scheduled to land/take off at airport } \Pi \]
\[ C \subseteq F \times F = \text{ set of ordered flight pairs } (i, j) \in F \times F \text{ such that there is a connection from } i \text{ to } j \]
\[ S_{it}^{\text{arr}} / S_{it}^{\text{dep}} = \begin{cases} 1 & \text{if flight } i \text{ is scheduled to land/take off no earlier than period } t \\ 0 & \text{otherwise} \end{cases} \]
\[ t_{ij}^{\text{min}} / t_{ij}^{\text{max}} = \text{minimum/maximum connection time between flight } i \text{ and flight } j \quad \forall (i, j) \in C \]

A connection refers to any pair of flights between which a minimum and a maximum time must be maintained to enable an aircraft, passengers, or a crew to connect. Note that the set of flights considered in the model may include flights that are not scheduled to land or take off at the airport \( \Pi \) where the scheduling interventions are applied, i.e., \( F_{\text{arr}} \cup F_{\text{dep}} \) may be smaller than \( F \). This arises from the need to maintain feasible connections in a network of airports.

**Variables.**

\[ w_{it}^{\text{arr}} / w_{it}^{\text{dep}} = \begin{cases} 1 & \text{if flight } i \text{ is rescheduled to land/take off no earlier than period } t \\ 0 & \text{otherwise} \end{cases} \]
\[ u_i = \text{displacement (positive or negative) of flight } i, \text{ as number of 15-minute periods} \]
\[ \lambda_t^{\text{arr}} / \lambda_t^{\text{dep}} = \text{number of arrivals/departures scheduled at airport } \Pi \text{ during period } t, \text{ after rescheduling} \]

By convention, we assume that \( w_{it+1}^{\text{arr}} = w_{it+1}^{\text{dep}} = 0, \forall i \in F \).

**Objective.** The model minimizes, first, the largest schedule displacement that any flight will sustain, denoted by \( \delta \) (Equation (1)), and, second, the total schedule displacement, denoted by \( \Delta_0 \).
\[ \delta = \max_{i \in F} |u_i| \]  
\[ \Delta_0 = \sum_{i \in F} |u_i| \] (Equation (2)).

*Scheduling Constraints.* For notational ease, a parameter \( \kappa \) refers either to the “arr” or “dep” superscript of the inputs and variables defined above.

\[ w_{it}^\kappa \geq w_{i,t+1}^\kappa \quad \forall i \in F, \forall \kappa \in \{\text{arr, dep}\}, \forall t \in T \] (3)

\[ w_{i1}^\kappa = 1 \quad \forall i \in F, \forall \kappa \in \{\text{arr, dep}\} \] (4)

\[ \sum_{t \in T} (w_{it}^\kappa - S_{it}^\kappa) = u_i \quad \forall i \in F, \forall \kappa \in \{\text{arr, dep}\} \] (5)

\[ \sum_{t \in T} (w_{jt}^{\text{dep}} - w_{it}^{\text{arr}}) \geq t_{ij}^{\min} \quad \forall (i, j) \in C \] (6)

\[ \sum_{t \in T} (w_{jt}^{\text{dep}} - w_{it}^{\text{arr}}) \leq t_{ij}^{\max} \quad \forall (i, j) \in C \] (7)

\[ \sum_{i \in F^k} (w_{it}^\kappa - w_{i,t+1}^\kappa) = \lambda_t^\kappa \quad \forall t \in T, \forall \kappa \in \{\text{arr, dep}\} \] (8)

Constraint (3) ensures that \( w^{\text{arr}} \) and \( w^{\text{dep}} \) are non-increasing in \( t \). Constraint (4) ensures that no flight is eliminated. Constraint (5) defines flight displacement as the difference between rescheduled time and original scheduled time, and ensures that the scheduled block-times are left unchanged. Constraints (6) and (7) maintain all connections by ensuring that connection times lie within the specified range. Constraint (8) defines the aggregate schedule of flights (i.e., the vector of the number of scheduled arrivals and the vector of the number of scheduled departures per time period), which will determine airport on-time performance. We summarize next the model of airport congestion that quantifies arrival and departure queue lengths as a function of the schedule of flights (i.e., \( \lambda_t^{\text{arr}} \) and \( \lambda_t^{\text{dep}} \)).

Arrival and departure queues are quantified by means of stochastic \( M(t)/E_3(t)/1 \) queuing systems, i.e., the demand processes are modeled as time-varying Poisson processes, and the service processes are modeled as time-varying Erlang processes of order 3. For each period \( t \), the arrival and departure demand rates are determined by flight schedules, i.e., they are equal to \( \lambda_t^{\text{arr}} \) and \( \lambda_t^{\text{dep}} \), respectively. Service rates are constrained by airport capacity. To capture the endogeneity of the service rates with respect to airport capacity and flight schedules, a control of capacity utilization procedures is integrated into the Stochastic Queuing Model of congestion. It is formulated as a finite-horizon Dynamic Programming model, that minimizes congestion costs for a given schedule of flights (Jacquillat et al., accepted for publication, 2015). At the beginning of each 15-minute period, the control selects the runway configuration and the balance of arrival and departure service rates
for that period, under capacity constraints, as a function of observed arrival and departure queue
lengths, the runway configuration in use, and wind and weather conditions. The combination of the
Stochastic Queuing Model and the Dynamic Programming model of capacity utilization provides
an integrated model of airport congestion that is computationally efficient and that approximates
well the magnitude and dynamics of delays at busy US airports (Jacquillat and Odoni, 2015). This
integrated model quantifies the relationship $q$ between flight schedules and flight delays, represented
as follows (where $A_t$ and $D_t$ denote the random variables that represent the arrival and departure
queue lengths at the end of period $t$):

$$ q : (\lambda_1^{\text{arr}}, \ldots, \lambda_T^{\text{arr}}, \lambda_1^{\text{dep}}, \ldots, \lambda_T^{\text{dep}}) \mapsto (A_1, \ldots, A_T, D_1, \ldots, D_T) \ (9) $$

Based on this relationship, the ICUSM aims to ensure that, at any time of the day, the expected
arrival and departure queue lengths do not exceed the prespecified limits, denoted by $A_{\text{MAX}}$ and
$D_{\text{MAX}}$, respectively (Constraints (10) and (11) below).

$$ E(A_t) \leq A_{\text{MAX}} \quad \forall t \in T \quad (10) $$

$$ E(D_t) \leq D_{\text{MAX}} \quad \forall t \in T \quad (11) $$

However, these constraints cannot be directly formulated into the Integer Programming schedul-
ing model described above, as the queuing dynamics (Equation (9)) depend nonlinerly on the
schedule of flights, hence on the model’s decision variables. The solution of the ICUSM therefore
relies on an algorithm that iterates between the Integer Programming model of scheduling inter-
ventions, the Dynamic Programming model of capacity utilization, and the Stochastic Queuing
Model of airport congestion outlined above, until it converges to the optimal value of the schedule
displacement. For any given expected queue length targets, this algorithm terminates in 10-15 iter-
ations and in 90 minutes to several hours on a modern computer, depending on model parameters.
For more details, we refer the reader to Jacquillat and Odoni (accepted for publication, 2015).

This solution approach integrates the on-time performance constraints (Constraints (10) and (11))
into a model of scheduling interventions, given the endogeneity of airport operating procedures and
airport congestion. Note that it can be applied to other formulations of the model of scheduling
interventions, as shall be done in this paper. In the remainder of this paper, any reference to on-
time performance constraints (such as Constraints (10) and (11)) will involve an iterative algorithm
adapted from the one outlined above.

2.2. Limitations

The ICUSM provides a modeling framework for optimizing scheduling interventions in order to
mitigate airport congestion. Its implementation quantifies the trade-off between schedule displace-
ment (i.e., $\delta$ and $\Delta_0$) and peak expected queue length limits (i.e., $A_{\text{MAX}}$ and $D_{\text{MAX}}$). However, it
suffers from the following two limitations:

1. The ICUSM does not account for inter-airline equity considerations. Its two-stage lexicographic formulation (characterized by Equations (1) and (2)) ensures equity at the flight level, i.e., no flight is disproportionately displaced. However, it does not ensure equity at the airline level. In turn, its solution may penalize some airlines disproportionately.

2. The ICUSM assumes that all flights are equally inconvenient to reschedule. Even though this “a flight is a flight” paradigm is standard in the airline industry, it may not be consistent with airlines’ scheduling preferences and with underlying passenger demand patterns.

3. Multi-criteria Modeling Architecture

We now present a modeling approach that builds upon the ICUSM to account for inter-airline equity and airlines’ preferences regarding which flights to reschedule. The general modeling structure, the types of decisions that are made, and the scheduling and network connectivity constraints are identical to those in the ICUSM, but the main difference lies in the objectives of scheduling interventions.

In addition to the notations introduced in Section 2, we partition the set of flights scheduled at airport Π, i.e., $F_{arr} \cup F_{dep}$, into subsets of flights scheduled by the different airlines. We also introduce a parameter $v$ to characterize “flight valuations”, reflecting airlines’ preferences. Flights with lower valuations can be thought of as less “costly” flights to reschedule, or as the flights that exhibit larger timetabling flexibility.

$$\mathcal{A} = \text{set of airlines, indexed by } \{1, ..., A\}$$
$$\mathcal{F}_a = \text{set of flights scheduled by airline } a \text{ at airport } \Pi$$
$$v_i = \text{valuation of flight } i$$

With these notations, we have: $\mathcal{F}_{a_1} \cap \mathcal{F}_{a_2} = \emptyset, \forall a_1 \neq a_2 \in \mathcal{A}$ and $\bigcup_{a \in \mathcal{A}} \mathcal{F}_a = \mathcal{F}_{arr} \cup \mathcal{F}_{dep}$.

In this section, we assume that flight valuations data $v$ are known to the central decision-maker. We propose performance attributes for scheduling interventions and develop a modeling architecture to characterize the associated trade space. In Section 5, we design non-monetary mechanisms that formalize the process through which the airlines can provide their scheduling inputs, including flight valuations data.

3.1. Performance Attributes

We consider the following three performance attributes of scheduling interventions: efficiency, inter-airline equity, and on-time performance. Efficiency and on-time performance extend the notions of schedule displacement and expected queue length limits, respectively, that are considered in the ICUSM, while the notion of equity is added to this framework.
Efficiency. This refers to the mechanism’s ability to meet airline scheduling preferences. Since no flight is eliminated, efficiency is measured by the displacement from the schedule of flights provided by the airlines. We consider the following two efficiency objectives. First, we define min-max efficiency as the largest displacement sustained by any flight in the scheduling interventions. As in Section 2, we denote it by $\delta$. Second, we define weighted efficiency as the sum of weighted schedule displacements sustained by all flights in the scheduling interventions, and we denote it by $\Delta$. Weighted efficiency generalizes the total displacement $\Delta_0$ considered in the ICUSM in a way that accounts for flight valuations. Directionally, maximizing efficiency involves minimizing $\delta$ and/or $\Delta$.

$$\delta = \max_{i \in F} |u_i| \implies \min \delta \tag{12}$$

$$\Delta = \sum_{i \in F} v_i |u_i| \implies \min \Delta \tag{13}$$

Inter-airline Equity. This refers to the mechanism’s ability to balance schedule displacement fairly among the airlines. We describe each airline’s disutility as the weighted average of per-flight displacements, denoted by $\sigma_a$. Perfect equity is achieved when the weighted sum of per-flight displacements borne by any airline is proportional to its number of flights scheduled at airport $\Pi$. In order to maximize inter-airline equity, we minimize airline disutilities lexicographically, i.e., we first minimize the largest weighted per-flight displacement borne by any airline, then the second-largest, etc. This extends the min-max equity formulation (Bertsimas et al., 2011a, 2012), and has been applied in other resource allocation problems (Klein et al., 1992; Ogryczak et al., 2005; Sun, 2011).

$$\sigma_a = \frac{1}{|F_a|} \sum_{i \in F_a} v_i |u_i|, \forall a \in A \implies \text{lex min } \sigma \tag{14}$$

We denote the largest airline disutility by $\Phi$:

$$\Phi = \max_{a \in A} \sigma_a \tag{15}$$

On-time performance. This refers to the mechanism’s ability to mitigate airport congestion. We quantify on-time performance by a non-decreasing function of the arrival and departure queue lengths $A_t$ and $D_t$ (which depend on the schedule of flights according to Equation (9)), denoted by $g(A_1, ..., A_T, D_1, ..., D_T)$. Examples of such functions include the peak expected arrival and departure queue lengths, the total delay experienced over a day of operations, the 95th percentile of the peak arrival and departure queue lengths, etc. Maximizing on-time performance involves minimizing the function $g$.

$$\min \{g (A_1, ..., A_T, D_1, ..., D_T)\} \tag{16}$$

The optimization of scheduling interventions is a multi-objective optimization problem. First,
each of these three performance attributes comprises several dimensions (e.g., minimizing min-max efficiency vs. weighted efficiency; minimizing the largest airline disutility vs. variations in airlines’ utilities for equity; minimizing arrival vs. departure delays for on-time performance). Moreover, there exists a trade-off between efficiency and on-time performance, quantified by the ICUSM: the larger the schedule displacement, the larger the potential delay reductions (up to a limit). Finally, there may be, for given on-time performance objectives, a trade-off between efficiency and equity (Bertsimas et al., 2011a, 2012).

3.2. Lexicographic Modeling Approach

We characterize the trade space between efficiency, equity, and on-time performance in airport scheduling interventions. In order to provide a transparent and optimal characterization of this trade space, we aim to find its Pareto frontier, i.e., the set of solutions such that, compared with any solution in this set, no other solution could improve at least one of the three objectives without worsening the others. This representation of the trade space is flexible enough to be used by system managers and policy makers to select the most appropriate level of compromise between these objectives. To this end, we develop a lexicographic optimization approach that (i) fixes on-time performance targets; (ii) maximizes efficiency under on-time performance targets; and (iii) maximizes equity under on-time performance and efficiency targets.

First, we quantify on-time performance by the peak expected arrival and departure queue lengths, i.e. 
\[ g(A_1,\ldots,A_T,D_1,\ldots,D_T) = (\max_{t \in T} E(A_t), \max_{t \in T} E(D_t)) \]. It is motivated by the objective of controlling the largest delays experienced over the day. Corresponding on-time performance constraints are identical to those in the ICUSM (Constraints (10) and (11)). We then aim to find the “best” schedule (in terms of efficiency and equity) that meets these constraints.

Second, we determine the schedule of flights that maximizes efficiency, subject to scheduling constraints, network connectivity constraints, and on-time performance constraints. We formulate the efficiency-maximizing problem by lexicographically maximizing, first, min-max efficiency \( \delta \), and, second, weighted efficiency \( \Delta \). This is motivated by the objective of avoiding large flight displacements, and consistent with the literature on this topic (Pyrgiotis and Odoni, Articles in advance, 2015; Jacquillat and Odoni, accepted for publication, 2015). This is expressed in Problems P1 and P2 described below:

**P1.** We minimize min-max efficiency metric \( \delta \), subject to scheduling, network connectivity and on-time performance constraints. We denote by \( \delta^* \) its optimal value.

\[
\min \quad \delta \quad \text{(Equation (12))} \\
\text{s.t.} \quad \text{Scheduling and network connectivity constraints: (3) to (8)} \\
\quad \text{On-time performance constraints: (9) to (11)}
\]
We minimize weighted efficiency metric $\Delta$, subject to scheduling, network connectivity and on-time performance constraints, and subject to the constraint that no flight may be displaced by more than $\delta^*$. We denote by $\Delta^*$ its optimal value.

$$\min \Delta \text{ (Equation (13))}$$

subject to:

- Scheduling and network connectivity constraints: (3) to (8)
- On-time performance constraints: (9) to (11)
- Min-max efficiency objectives: $|u_i| \leq \delta^*, \forall i \in \mathcal{F}$

Third, we maximize inter-airline equity, subject to scheduling constraints, network connectivity constraints, on-time performance constraints, and efficiency targets. This is formulated in the class of problems $P3(\rho)$ described below:

$P3(\rho)$. We fix efficiency targets, and we lexicographically minimize airline disutilities, subject to scheduling, network connectivity, on-time performance, and efficiency constraints. We characterize the trade space between efficiency and equity by varying the efficiency target. Specifically, we impose that min-max efficiency must be optimal (i.e., no flight may be rescheduled by more than $\delta^*$) and we denote by $\rho \in [0, \infty)$ the relative loss in weighted efficiency that is allowed (i.e., the weighted displacement must not exceed $(1+\rho)\Delta^*$). When $\rho = \infty$, we only maximize equity (without any weighted efficiency consideration). When $\rho = 0$, we maximize equity, under optimal efficiency.

$$\text{lex min } \sigma \text{ (Equation (14))}$$

subject to:

- Scheduling and network connectivity constraints: (3) to (8)
- On-time performance constraints: (9) to (11)
- Min-max efficiency objectives: $|u_i| \leq \delta^*, \forall i \in \mathcal{F}$
- Weighted efficiency objectives: $\sum_{i \in \mathcal{F}} v_i |u_i| \leq (1 + \rho) \Delta^*$

Problems $P1$, $P2$, and $P3(\rho)$ combined determine the Pareto frontier of the trade space between efficiency, equity, and on-time performance in scheduling interventions. First, variations in the on-time performance targets $A_{\text{MAX}}$ and $D_{\text{MAX}}$ quantify the trade-off between the costs of scheduling interventions (in terms of inefficiency and inequity) and delay reductions. Second, for any on-time performance targets $A_{\text{MAX}}$ and $D_{\text{MAX}}$, varying the parameter $\rho$ quantifies the potential trade-off between weighted efficiency and inter-airline equity (achieved while keeping the min-max efficiency at its optimal value).

Figure 1 illustrates our approach to maximizing weighted efficiency and inter-airline equity, for given on-time performance targets $A_{\text{MAX}}$ and $D_{\text{MAX}}$, and the optimal value of min-max efficiency $\delta^*$. Specifically, it shows hypothetical variations in three airlines’ disutilities ($\sigma_1$, $\sigma_2$, and $\sigma_3$) as a
function of the weighted efficiency target $\Delta = (1 + \rho)\Delta^*$. By construction, the region on the left side of $\Delta^*$ is infeasible, i.e., the weighted schedule displacement must be at least $\Delta^*$. Moreover, the largest airline disutility $\Phi$ is a non-increasing function of the value of weighted efficiency $\Delta$ (i.e., of $\rho$). Note that the other airlines’ utilities (here, $\sigma_2$ and $\sigma_3$) may increase or decrease as $\Phi$ is reduced. As the largest airline disutility $\Phi$ attains its optimal value, the second-largest disutility may still be larger than its optimal value. In this case, further increases in $\rho$ may yield further improvements in the lexicographic minimization of airline disutilities. Optimal equity is attained when the largest, second largest, third largest, etc., airline disutilities have all reached their optimal values (i.e., the values that would be obtained without any efficiency consideration, or with $\rho = \infty$). Note that Figure 1 shows an instance where the order of airline disutilities remains identical for all values of $\rho$ (i.e., in this case, $\sigma_1(\rho) > \sigma_2(\rho) > \sigma_3(\rho), \forall \rho \geq 0$), but this need not be the case (i.e., the curves may intersect). We denote by $\sigma^*(\rho)$ the equity-maximizing vector of airline per-flight displacements, as a function of $\rho$, and $\Phi^*(\rho) = \max_{a \in A} \sigma^*_a(\rho)$. We denote by $\Delta^{eq}$ the smallest equity-maximizing value of $\Delta$, and by $\rho^*$ the minimum loss in weighted efficiency required to attain optimal equity (i.e., $\Delta^{eq} = (1 + \rho^*)\Delta^*$). With these notations, the “price of efficiency” and the “price of equity” will be characterized by $P_{eff} = \frac{\Phi^*(0) - \Phi^*(\infty)}{\Phi^*(\infty)}$, and by $P_{eq} = \frac{\Delta^{eq} - \Delta^*}{\Delta^*} = \rho^*$, respectively.

Figure 1: A schematic trade space between weighted efficiency and equity

3.3. Solution Architecture

As discussed in Section 2, the on-time performance constraints are not linear. Solving Problems $P_1$, $P_2$, and $P_3(\rho)$ thus requires a solution algorithm such as the one outlined in Section 2. To solve Problem $P_1$, we update iteratively a lower bound of the optimal maximum flight displacement $\delta^*$, i.e., we increase the value of the maximum flight displacement from 0 15-minute period to 1 period, then 2 periods, until a feasible schedule that meets the on-time performance targets is
found. To solve Problem $P2$, we iteratively update an upper bound $\Delta$ and a lower bound $\Delta$ of the optimal weighted displacement $\Delta^*$, by dichotomy. At each iteration, we consider a value of $\Delta = \frac{\Delta + \Delta}{2}$, and we update $\Delta$ (resp. $\Delta$) to $\frac{\Delta + \Delta}{2}$ if the resulting delay estimates meet (resp. do not meet) the on-time performance constraints. We repeat the process until the following stopping criteria is reached: $\frac{\Delta - \Delta}{2} \leq \varepsilon$. This ensures that the schedule displacement obtained is within $\varepsilon$ of the optimal schedule displacement. We use a value of 1% for $\varepsilon$. These algorithms are adopted from Jacquillat and Odoni (accepted for publication, 2015).

When solving the equity-maximizing problem (Problem $P3(\rho)$), one candidate approach is to design a similar iterative algorithm. This would consist of iteratively updating lower and upper bounds of the optimal value of airlines’ disutilities $\sigma_a$ until convergence. However, the computational requirements of the solution algorithm described in Section 2 prevent it from being applied repeatedly for several airlines, for several values of the parameter $\rho$, and with different sets of inputs. For this reason, we develop an alternative approach that approximates Problem $P3(\rho)$ while ensuring computational tractability (by reducing its computational requirements from several hours to a few minutes for each of the hundreds of runs we perform).

Specifically, we replace the on-time performance constraints (Constraints (10) and (11)) by scheduling limit constraints (Constraints (17) and (18), defined below). These constraints ensure that, for any period $t$, the number of scheduled arrivals and departures does not exceed limits denoted by $\lambda^\text{arr}_t$ and $\lambda^\text{dep}_t$, respectively. We refer to these constraints as “simplified on-time performance constraints”.

$$\lambda^\text{arr}_t \leq \lambda^\text{arr}_t, \forall t \in \mathcal{T}$$  \hspace{1cm} (17)

$$\lambda^\text{dep}_t \leq \lambda^\text{dep}_t, \forall t \in \mathcal{T}$$  \hspace{1cm} (18)

The resulting model is formulated below, and we refer to it as $\hat{P3}(\rho)$:

$$\text{lex min } \sigma \text{ (Equation (14))}$$

s.t. Scheduling and network connectivity constraints: (3) to (8)

Simplified on-time performance constraints: (17) and (18)

Min-max efficiency objectives: $|u_i| \leq \delta^*, \forall i \in \mathcal{F}$

Weighted efficiency objectives: $\sum_{i \in \mathcal{F}} v_i |u_i| \leq (1 + \rho) \Delta^*$

Unlike Problem $P3(\rho)$, Problem $\hat{P3}(\rho)$ is an Integer Programming model and can be solved directly using a commercial solver. Its solution is substantially faster than that of Problem $P3(\rho)$, which required iterating 10-15 times between an Integer Program, a Dynamic Program, and a Stochastic Queuing Model. The main challenge lies in setting appropriate values of the scheduling
limits $\hat{\lambda}_{\text{arr}}$ and $\hat{\lambda}_{\text{dep}}$. If set too high, they would not enable the resulting arrival and departure queue lengths to meet the on-time performance targets $A_{\text{MAX}}$ and $D_{\text{MAX}}$, respectively. If set too low, they may not minimize the displacement impact on airline schedules of flights. In this paper, we set the scheduling limits $\hat{\lambda}_{\text{arr}}$ and $\hat{\lambda}_{\text{dep}}$ equal to the aggregate number of scheduled arrivals and departures obtained by solving Problem $\text{P2}$. In other words, we first determine the efficiency-maximizing schedule of flights. We then look for flight schedules that achieve the same aggregate schedule (but not necessarily the same schedule for each individual flight), while yielding a Pareto-optimal solution to the trade-off between weighted efficiency and equity. Here we define aggregate schedule as the vector of the number of scheduled departures and the number of scheduled arrivals per time period.

By construction, the schedule obtained through this computationally efficient approach meets the delay reduction constraints (10) and (11). On the other hand, Constraints (17) and (18) could be more restrictive than Constraints (10) and (11), so this solution approach may yield a sub-optimal solution. The choice of this approach is motivated by four factors. First, the approach yields improvements in inter-airline equity without sacrificing other objectives, as compared to existing approaches. Second, it starts with the scheduling inputs provided by the airlines to determine the aggregate schedule, which can thus exhibit some peaks and valleys in accordance with airline scheduling preferences and passenger demand. Third, we expect the errors of our solution approach to be of second order. Our computational results reported in Section 6 indeed show that it leads to very high equity. Finally, its reliance on the aggregate schedule of flights (instead of individual flight schedules and expected delay reductions) makes this approach easily communicable and implementable.

Our full solution architecture is shown in Figure 2. It takes as inputs scheduling data, connections data, and flight valuation data, as well as on-time performance targets $A_{\text{MAX}}$ and $D_{\text{MAX}}$ set by the central decision-maker. First, we successively solve Problems $\text{P1}$ and $\text{P2}$, and we store the optimal efficiency values ($\delta^*$ and $\Delta^*$) and the aggregate schedule ($\hat{\lambda}_{\text{arr}}$ and $\hat{\lambda}_{\text{dep}}$). Second, we solve Problems $\hat{\text{P3}}(\rho)$ to determine the Pareto frontier of the trade space between weighted efficiency and equity to achieve this aggregate schedule (i.e., $\hat{\lambda}_{\text{arr}}$ and $\hat{\lambda}_{\text{dep}}$). We start by maximizing equity with no weighted efficiency constraint ($\rho = \infty$). We then maximize equity under optimal weighted efficiency ($\rho = 0$), and we relax progressively the weighted efficiency requirements by increasing $\rho$ in increments of 0.001, until optimal equity is reached. We use the following stopping criteria: \[
\frac{\sigma_a^*(\rho) - \sigma_a^*(\infty)}{\sigma_a^*(\infty)} \leq \varepsilon, \forall a \in A, \text{i.e., the algorithm terminates when all airlines’ disutilities are within } \varepsilon \text{ of their equity-maximizing values. We use here a value of 1% for } \varepsilon. \] This algorithm characterizes, for any pair of on-time performance targets $A_{\text{MAX}}$ and $D_{\text{MAX}}$, (i) the efficiency-maximizing schedule of flights, and (ii) feasible flight schedules that achieve the same aggregate schedule and yield different Pareto-optimal solutions to the trade-off between weighted efficiency and equity.

As mentioned before, we assumed thus far that the scheduling, connections and flight valuations
data are known to the decision-maker with accuracy. In reality, that is not the case. We thus design mechanisms that formalize the process through which the airlines can provide their inputs (i.e., flight schedules, connections and flight valuations data) in Section 5. Before this, we provide in the next section a theoretical discussion on the conditions under which efficiency and equity can be jointly maximized.

4. A Theoretical Discussion on Inter-airline Equity

We first show that, under some conditions on the scheduling inputs provided by the airlines, efficiency and equity can be jointly optimized. We consider in Section 4.1 the case where no network
connections need to be maintained and all flights are equally valued, and we identify conditions on flight schedules that enable the joint optimization of efficiency and equity. We then discuss in Section 4.2 the factors that may violate these conditions, and thus create a trade-off between efficiency and equity in scheduling interventions.

4.1. Cases of Joint Maximization of Efficiency and Equity

We assume in this section that no connections need to be maintained (i.e., \( C = \emptyset \)) and that all flights are equally valued (i.e., \( v_i = 1, \forall i \in \mathcal{F} \)). With \( C = \emptyset \), the problems of rescheduling arriving flights and departing flights can be treated separately. For simplicity of the exposition, we focus on the case of \( \delta^* = 1 \) period, which is also the most common case encountered with real-world data (see Section 6). We consider the following problems of efficiency maximization (EFF) and equity maximization (EQ), defined below, subject to the constraint that no more than \( \lambda^*_t \) flights may be scheduled during any period \( t \).

\[
\begin{align*}
\min \quad & \sum_{i \in \mathcal{F}} |u_i| \quad \text{(EFF)} \\
\text{s.t.} \quad & w_{it} \geq w_{i,t+1}, \forall i \in \mathcal{F}, \forall t \in \mathcal{T} \\
& w_{it} = 1, \forall i \in \mathcal{F} \\
& \sum_{t \in \mathcal{T}} (w_{it} - S_{it}) = u_i, \forall i \in \mathcal{F} \\
& \sum_{i \in \mathcal{F}} (w_{i,t+1} - w_{i,t}) \leq \lambda^*_t, \forall t \in \mathcal{T} \\
& |u_i| \leq 1, \forall i \in \mathcal{F}
\end{align*}
\]

\[
\begin{align*}
\min \quad & \left( \frac{1}{|\mathcal{F}|} \sum_{i \in \mathcal{F}_a} |u_i| \right)_{a \in \mathcal{A}} \quad \text{(EQ)} \\
\text{s.t.} \quad & w_{it} \geq w_{i,t+1}, \forall i \in \mathcal{F}, \forall t \in \mathcal{T} \\
& w_{it} = 1, \forall i \in \mathcal{F} \\
& \sum_{t \in \mathcal{T}} (w_{it} - S_{it}) = u_i, \forall i \in \mathcal{F} \\
& \sum_{i \in \mathcal{F}} (w_{i,t+1} - w_{i,t}) \leq \lambda^*_t, \forall t \in \mathcal{T} \\
& |u_i| \leq 1, \forall i \in \mathcal{F}
\end{align*}
\]

In the remainder of this section, we denote by \( \mathcal{D}_t \) the set of flights scheduled during period \( t \) before the scheduling interventions, i.e., \( \mathcal{D}_t = \{ i \in \mathcal{F} | S_{it} = 1 \land S_{i,t+1} = 0 \} \). By convention, we assume that \( \mathcal{D}_0 = \mathcal{D}_{t+1} = \emptyset \) and \( \hat{\lambda}_0 = \hat{\lambda}_{t+1} = 0 \). We also denote the positive part of any number \( x \) by \( x^+ = \max(x, 0) \).

Proposition 1 shows that efficiency and equity can be jointly maximized if the number of flights scheduled over any set of three consecutive time periods is lower than the total number of flights that can be scheduled over the same three periods. In that case, the scheduling interventions in the periods with more than \( \hat{\lambda}_t \) flights scheduled can be treated independently and the problem can thus be reduced to a series of one-period problems.

**Proposition 1.** If \( \sum_{t=1}^{t+1} |\mathcal{D}_t| \leq \sum_{t=1}^{t+1} \hat{\lambda}_t, \forall t \in \mathcal{T} \), then there exists a solution that simultaneously solves (EFF) and (EQ).

**Proof.** Any feasible solution has to displace at least \( (|\mathcal{D}_t| - \hat{\lambda}_t)^+ \) flights in every period \( t \), so \( \Delta^* \geq \sum_{t \in \mathcal{T}} (|\mathcal{D}_t| - \hat{\lambda}_t)^+ \). We first construct a feasible solution that reschedules exactly \( \sum_{t \in \mathcal{T}} (|\mathcal{D}_t| - \hat{\lambda}_t)^+ \) flights. To do so, for any period \( t \) such that \( |\mathcal{D}_t| > \hat{\lambda}_t \), we reschedule flights first to the preceding period (i.e., period \( t-1 \)), up to capacity, and then to the following period (i.e., period \( t+1 \)). Specifically, we select a subset \( \mathcal{K}_t^- \subset \mathcal{D}_t \) such that \( |\mathcal{K}_t^-| = \min \left\{ (\hat{\lambda}_{t-1} - |\mathcal{D}_{t-1}|)^+, (|\mathcal{D}_t| - \hat{\lambda}_t)^+ \right\} \).
Note that $K_t^-$ is not uniquely determined, but we can choose any subset of $D_t$ that verifies this property. We then select a subset $K_t^+ \subset D_t \setminus K_t^-$ such that $|K_t^+| = \left( |D_t| - \lambda_t - |K_t^-| \right)^+$. Like $K_t^-$, $K_t^+$ is not uniquely determined, but can be any set that verifies this property. We define $u^{\text{eff}}$ as follows: $u^{\text{eff}}_i = -1, \forall i \in K_t^-$ and $u^{\text{eff}}_i = +1, \forall i \in K_t^+$. We define $w^{\text{eff}}$ accordingly (based on the constraints of (EFF)). Under the assumption of the proposition, it is easy to see that $(w^{\text{eff}}, u^{\text{eff}})$ is a feasible solution. Moreover, it verifies $\sum_{i \in F} |u^{\text{eff}}_i| = \sum_{t \in T} \left( |D_t| - \lambda_t \right)^+$. Therefore, it solves (EFF) and $\Delta^* = \sum_{t \in T} \left( |D_t| - \lambda_t \right)^+$.

We now denote by $(w^{\text{eq}}, u^{\text{eq}})$ an optimal solution of (EQ). If $\sum_{i \in F} |u^{\text{eq}}_i| = \Delta^*$, then $u^{\text{eq}}$ also solves (EFF). We now assume that $\sum_{i \in F} |u^{\text{eq}}_i| > \Delta^*$. For each $t \in T$, we define the following set: $I_t = \{ i \in D_t | u^{\text{eq}}_i = 1 \}$. We have $|I_t| \geq \left( |D_t| - \lambda_t \right)^+, \forall t \in T$ (otherwise, $u^{\text{eq}}$ would not be a feasible solution of (EQ)). We can construct a set $J_t \subseteq I_t$ such that $|J_t| = \left( |D_t| - \lambda_t \right)^+$ for all $t \in T$. As with $K_t^-$ and $K_t^+$ earlier, $J_t$ is not uniquely determined, but we can choose any subset of $I_t$ that verifies this property. Let $J$ be defined by $J = \cup_{t \in T} J_t$. We construct a solution $u^*$ as follows: $u^*_i = 0, \forall i \notin J$ and $u^*_i = u^{\text{eq}}_i, \forall i \in J$. We define $w^*$ accordingly. By construction, for each period $t$ such that $|D_t| > \lambda_t$, this solution displaces exactly $|D_t| - \lambda_t$ flights, so $\sum_{i \in F} \left( w^*_i - w^{\text{eq}}_{i,t+1} \right) = \lambda_t$. Moreover, the number of flights rescheduled to the preceding and following time periods is smaller than under solution $(w^{\text{eq}}, u^{\text{eq}})$ (i.e., $\sum_{i \in F} \left( w^*_i,t-1 - w^{\text{eq}}_i,t \right) \leq \sum_{i \in F} \left( w^{\text{eq}}_i,t-1 - w^{\text{eq}}_i,t \right)$ and $\sum_{i \in F} \left( w^*_i,t+1 - w^{\text{eq}}_i,t+2 \right) \leq \sum_{i \in F} \left( w^{\text{eq}}_i,t+1 - w^{\text{eq}}_i,t+2 \right)$). Therefore, $(w^*, u^*)$ is a feasible solution of (EFF) and (EQ). Moreover, it verifies: $\sum_{i \in F} |u^*_i| = \Delta^*$, and $|u^*_i| \leq |u^{\text{eq}}_i|, \forall i \in F$. Therefore, $u^*$ solves (EFF) and (EQ).

Proposition 2 shows that efficiency and equity can be jointly maximized if each airline’s share of flights is identical across all periods. Specifically, we assume that the number of flights scheduled by each airline $a$ during each period $t$ is the product of an airline-related factor $\alpha_a$ and a period-related factor $\beta_t$. In that case, there is significant flexibility in terms of the airlines whose flights should be rescheduled, which enables equity-maximization at no efficiency loss.

**Proposition 2.** If there exist integers $(\alpha_a)_{a \in A}$ and $(\beta_t)_{t \in T}$ such that $|D_t \cap F_a| = \alpha_a \beta_t, \forall a \in A, t \in T$, then there exists a solution that simultaneously solves (EFF) and (EQ).

**Proof.** Proof We consider an optimal solution of (EFF), which we denote by $(w^{\text{eff}}, u^{\text{eff}})$. We denote by $X_t^+$ (resp. $X_t^-$) the number of flights that, under solution $(w^{\text{eff}}, u^{\text{eff}})$, are displaced from period $t$ to period $t+1$ (resp. $t-1$), i.e., $X_t^+ = |\{ i \in D_t | u^{\text{eff}}_i = +1 \}|$ (resp. $X_t^- = |\{ i \in D_t | u^{\text{eff}}_i = -1 \}|$). We also denote by $X_t$ the total number of flights displaced from period $t$, i.e., $X_t = X_t^- + X_t^+, \forall t \in T$. The optimal objective value function of (EFF) is $\Delta^* = \sum_{t=1}^n |u^{\text{eq}}_t| = \sum_{t \in T} (X_t^- + X_t^+) = \sum_{t \in T} X_t$. We aim to construct a solution $(w^*, u^*)$ that is feasible, efficient and equitable.
A sufficient condition for \((w^*, u^*)\) to be feasible and efficient is to ensure that, for each period \(t\), the number of flights rescheduled to \(t - 1\) and to \(t + 1\), respectively, under solution \((w^*, u^*)\) is equal to that under solution \((w^\text{eff}, u^\text{eff})\) for every period \(t\), that is \(|\{i \in D_t | u^*_i = -1\}| = X^+_t\) and \(|\{i \in D_t | u^*_i = +1\}| = X^-_t\), \(\forall t \in T\). Indeed, if this condition is verified, the aggregate schedule is identical under solutions \((w^\text{eff}, u^\text{eff})\) and \((w^*, u^*)\) (i.e., \(\sum_{i \in F} (w^\text{eff}_{it} - w^*_{it+1}) = \sum_{i \in F} (w^*_{it} - w^*_{it+1})\), \(\forall t \in T\)), so solution \((w^*, u^*)\) is feasible. Moreover, under this condition: \(\sum_{i \in D_t} |u^*_i| = X^-_t + X^+_t\), \(\forall t \in T\), and by summing over \(t\) we obtain: \(\sum_{t \in T} \sum_{i \in D_t} |u^*_i| = \sum_{t \in T} X_t\), i.e., \(\sum_{i \in F} |u^*_i| = \Delta^*\), so solution \((w^*, u^*)\) is efficient.

A sufficient condition for \((w^*, u^*)\) to be equitable is to ensure that the vector \(U\) defined by \(U_a = \sum_{i \in F_a} |u^*_i|\), \(\forall a \in A\) solves the following problem, denoted by \(P(\Delta^*)\):

\[
\begin{align*}
\text{lex min} & \quad \frac{U_a}{|F_a|}_{a \in A} \\
\text{s.t.} & \quad \sum_{a \in A} U_a \geq \Delta^* \\
& \quad U_a \geq 0, U_a \text{ integer}
\end{align*}
\]

We construct an optimal solution of Problem \(P(\Delta^*)\) in the appendix (Lemma 7), which we will use in this proof to construct a solution of (EQ). First, let us summarize how this solution of Problem \(P(\Delta^*)\) is constructed. We assume without loss of generality that the greatest common divisor (gcd) of \((\alpha_a)_{a \in A}\) is equal to 1. Note that \(|F_a| = \alpha_a \sum_{t \in T} \beta_t\), \(\forall a \in A\) and thus: \(\text{gcd}(|F_a|)_{a \in A} = 1\). We then have: \(\frac{|F_a|}{\text{gcd}(|F_a|)_{a \in A}} = \alpha_a, \forall a \in A\). We denote by \(N = \sum_{a \in A} \alpha_a\). Let \(\mathbb{1}\) denote the indicator (i.e., \(\sum_{i=1}^N \mathbb{1}(a_i = a) = \alpha_a, \forall a \in A\) and the \(|A|\)-dimensional vector \(U\) defined by \(U_a = q\alpha_a + \sum_{i=1}^r \mathbb{1}(a_i = a), \forall a \in A\) is an optimal solution of \(P(\Delta^*)\), where \(q\) and \(r\) denote the quotient and the remainder of the Euclidean division of \(\Delta^*\) by \(N\) (i.e., \(\Delta^* = qN + r\)). We denote by \(\Psi\) the sequence \(\Psi = (a_1, ..., a_N, a_1, ..., a_N, a_1, ..., a_r)\), where the full sequence \((a_1, ..., a_N)\) is repeated \(q\) times. By construction, \(\sum_{i=1}^{\Delta^*} \mathbb{1}(\Psi_i = a) = q\alpha_a + \sum_{i=1}^r \mathbb{1}(a_i = a), \forall a \in A\) and thus the vector \(U\) defined by \(U_a = \sum_{i=1}^{\Delta^*} \mathbb{1}(\Psi_i = a)\), \(\forall a \in A\) is an optimal solution of \(P(\Delta^*)\).

We now construct a solution \((w^*, u^*)\) that verifies (i) the sufficient conditions for feasibility and for efficiency maximization: \(\{|i \in D_t | u^*_i = -1\}| = X^+_t\), \(\forall t \in T\) and \(\{|i \in D_t | u^*_i = +1\}| = X^-_t\), \(\forall t \in T\), and (ii) the sufficient condition for equity maximization: \(\sum_{i \in F_a} |u^*_i| = \sum_{i=1}^{\Delta^*} \mathbb{1}(\Psi_i = a), \forall a \in A\). To do so, we construct a solution that displaces flights from the sequence of airlines \(\Psi\), i.e., a solution that displaces flights from airline \(\Psi_1\), ..., from airline \(\Psi_{X_1+1}\), ..., from airline \(\Psi_{X_1+X_2}\) in period 2, etc. (of course, each airline may be repeated several times in each sequence). We denote by \(y_t\) the number of flights displaced from periods between period 1 and period \(t - 1\) (both inclusive), i.e., \(y_t = \sum_{s=1}^{t-1} X_s\). Note that \(y_1 = 0\) and \(y_{T+1} = \sum_{s \in T} X_s = \Delta^*\). We denote by \(V_{at}\) the number of times airline \(a\) is repeated in the \(X_t\) indices between \(y_t + 1\) and
$y_{t+1}$ (both inclusive), i.e.: $V_{at} = \sum_{i=y_{t}+1}^{y_{t+1}} \mathbb{1}(\Psi_i = a), \forall a \in A$. Given the periodicity of the sequence $\Psi$, it is easy to see that $V_{at} \leq \alpha_a \beta_t$. We can thus define a set $J_{at} \subseteq D_t \cap F_a$ such that $|J_{at}| = V_{at}$.

As in the proof of Proposition 1, $J_{at}$ is not uniquely determined, but we can choose any subset of $D_t \cap F_a$ that verifies this property. We construct a solution that displaces the flights in the sets $J_{at}$ such that the number of flights rescheduled to period $t-1$ (resp. $t+1$) is equal to $X_t^-$ (resp. $X_t^+$). For each $t \in T$, we partition $\cup_{a \in A} J_{at}$ into two subsets $K_t^+$ and $K_t^-$ such that $|K_t^+| = X_t^+$ and $|K_t^-| = X_t^-$. We then define (i) $u_i^* = -1, \forall i \in K_t^-$, (ii) $u_i^* = +1, \forall i \in K_t^+$, (iii) $u_i^* = 0, \forall i \notin K_t^- \cup K_t^+$. We define $w^*$ accordingly (based on the constraints of (EFF) and (EQ)).

By construction, the solution $(w^*, u^*)$ satisfies the sufficient conditions for feasibility and efficiency maximization, so it solves Problem (EFF). Moreover, we have: $\sum_{i \in B \cap F_a} u_i^* - \sum_{t \in T} \sum_{i = y_{t} + 1}^{y_{t+1}} \mathbb{1}(\Psi_i = a)$, that is $\sum_{i \in F_a} u_i^* = \sum_{i = y_{t} + 1}^{y_{t+1}} \mathbb{1}(\Psi_i = a) = \sum_{i = 1}^{\Delta^*} \mathbb{1}(\Psi_i = a)$. Therefore, the solution $(w^*, u^*)$ solves Problem (EQ).

In summary, efficiency and equity can be jointly maximized if (i) no network connections need to be maintained, (ii) all flights are equally valued, and (iii) airline schedules of flights satisfy the conditions of Proposition 1 or Proposition 2 (or both). Under the conditions of Proposition 1, the imbalances between demand and capacity are small enough so no time period is such that some flights get displaced to that period and some other flights get displaced from that period. Under the conditions of Proposition 2, the schedules of flights of the different airlines exhibit the same intra-day variations. Even though these conditions are somewhat restrictive and are usually not exactly satisfied in practice, our computational experiments reported in Section 6 show that the insights derived in these two simple cases can be relevant and applicable in practical settings.

4.2. Instances of Efficiency/Equity Trade-off

Based on the discussion above, a trade-off between efficiency and equity can arise through (i) inter-airline variations in intra-day flight schedule patterns (let us refer to this factor simply by ‘differentiated flight schedules’), (ii) network connections, and (iii) intra-airline variations in flight valuations (let us refer to this factor simply by ‘differentiated flight valuations’).

We first provide an example that shows that weighted efficiency and equity may not be jointly maximized in the presence of differentiated airline schedules. Figure 3 shows a hypothetical unconstrained schedule in a 7-period case with 2 airlines and 26 flights per airline, and a simple capacity constraint that ensures that no more than 10 flights may be scheduled per period. Assume that all flights are valued equally and that there are no connections. Figure 3a (resp. Figure 3b) shows which flights are rescheduled to later or earlier times for an efficiency-maximizing solution (resp. an equity-maximizing solution). We assume that airline 1’s flights (one such efficiency-maximizing solution is shown in red) are concentrated at earlier periods, and airline 2’s flights (shown in green) are concentrated at later periods. Note that the conditions of either Proposition 1 or 2 are not satisfied here. The capacity constraint is only violated during period 5, when all flights scheduled are
airline 2’s flights. In this case, every equity-maximizing solution displaces 4 flights from airline 2 to later times, by 1 period each (one such equity-maximizing solution is shown as “+1” in Figure 3a). The resulting total displacement is equal to 4 periods, and the airline disutilities are equal to 0 for airline 1 and to 4/26 for airline 2. In contrast, every efficiency-maximizing solution displaces 3 flights from airline 1 and 3 flights from airline 2 to earlier times, by 1 period each (shown as “-1” in Figure 3b). The resulting total displacement is equal to 6 periods, and each airline’s disutility is equal to 3/26. In turn, the set of efficiency-maximizing solutions and the set of equity-maximizing solutions have no overlap.

![Figure 3: Trade-off between weighted efficiency and equity due to differentiated airline schedules](image)

We now provide an example that shows that weighted efficiency and equity may not be jointly optimized in the presence of network connections. Intuitively, if one airline’s network is significantly more connected than another airline’s, then the flights from the former airline are likely to be more difficult to reschedule. In turn, maximizing efficiency may involve assigning more displacement to the latter airline’s flights rather than the former’s, at some equity loss. Figure 4 shows such an example with 5 periods, 2 airlines with 13 flights each, and a capacity of 6 flights per period. Note that the conditions of both Propositions 1 and 2 would be verified in the absence of network connections. But airline 1’s network involves a number of connections, whereas airline 2’s network has no connections. We represent connections by dashed, gray “links” between flight pairs, and we assume that each connection requires a 2-period interval between the flights in the connection at a minimum. In this case, every efficiency-maximizing solution displaces 4 flights from airline 2 (the airline with no connections) by 1 period each. The resulting total displacement is equal to 4 periods, and the airline disutilities are equal to 0 for airline 1 and to 4/13 for airline 2. In contrast, every equity-maximizing solution displaces 3 flights from airline 1 and 3 flights from airline 2, by 1 period each. The resulting total displacement is equal to 6 periods, and each airline’s disutility is equal to 3/13. Again, the set of efficiency-maximizing solutions and the set of equity-maximizing solutions have no overlap.
Finally, we provide an example that shows that weighted efficiency and equity may not be jointly optimized in the presence of intra-airline variations in flight valuations. Figure 5 shows an example with 5 periods, 2 airlines with 10 flights each, and a capacity of 6 flights per period. Note that the conditions of both Propositions 1 and 2 would be verified under uniform flight valuations. But we assume that every flight, except the 6 flights scheduled by airline 2 in period 3, has a value equal to \( v_i = 1 \). Specifically, three of the 6 flights scheduled by airline 2 in period 3 have a value \( v_i = 0.1 \) each, and three others have a value \( v_i = 1.9 \) (the average value of airline 2’s flights is equal to 1). Every efficiency-maximizing solution displaces the three flights of value \( v_i = 0.1 \) and three flights of value \( v_i = 1 \). The optimal value of the weighted displacement is equal to 3.3 and the airline disutilities are equal to 0.3/10 for airline 1 and to 0.3/10 for airline 2. In contrast, every equity-maximizing solution displaces two flights of airline 1 and four flights of airline 2. The weighted displacement is equal to 4.2 and the airline disutilities are equal to 2/10 for airline 1 and to 2.2/10 for airline 2. Again, the set of efficiency-maximizing solutions and the set of equity-maximizing solutions have no overlap.

5. Mechanisms for Airport Scheduling Interventions

The models for scheduling interventions developed in this paper rely on scheduling data (i.e., \( F, \ F_{\text{arr}}, \ F_{\text{dep}}, \ F_o, \ S_{\text{arr}}, \) and \( S_{\text{dep}} \)), connections data (i.e., \( C, t_{\text{min}}, \) and \( t_{\text{max}} \)), and flight valuations data (i.e., \( v \)). However, in the current scheduling environment at US airports, flights’ timetabling flexibility is private information of the airlines and is unknown to the central decision-maker. In this section, we develop several mechanisms that specify the scheduling inputs provided by the airlines and the subsequent scheduling interventions optimized by the central decision-maker. In order to be implementable, we ensure that the considered mechanisms are non-monetary, transparent, and computationally tractable.

First, we define below a Baseline Scheduling Mechanism (BSM). The BSM is the mechanism
Mechanism 1 Baseline Scheduling Mechanism (BSM)

The airlines provide their preferred schedules of flights \( \rightarrow F, F^{\text{arr}}, F^{\text{dep}}, F_a, S^{\text{arr}}, S^{\text{dep}} \)

The airlines provide the network connections to be maintained \( \rightarrow C, t^{\text{min}}, t^{\text{max}} \)

The central decision-maker assumes uniform flight valuations \( \rightarrow v_i = 1, \forall i \in F \)

The central decision-maker solves Problems \( P1 \) and \( P2 \)

Our second mechanism is similar to the BSM but considers inter-airline equity objectives in the optimization of scheduling interventions. As in the BSM, the airlines submit only their preferred schedule of flights and the set of flight connections to be maintained, and the central decision-maker assumes uniform flight valuations. However, scheduling interventions are optimized with respect to efficiency, equity, and on-time performance objectives (i.e., we solve Problems \( P1, P2, \) and \( P3(\rho) \)). We name this mechanism the *Equitable Scheduling Mechanism (ESM)*.

Mechanism 2 Equitable Scheduling Mechanism (ESM)

The airlines provide their preferred schedules of flights \( \rightarrow F, F^{\text{arr}}, F^{\text{dep}}, F_a, S^{\text{arr}}, S^{\text{dep}} \)

The airlines provide the network connections to be maintained \( \rightarrow C, t^{\text{min}}, t^{\text{max}} \)

The central decision-maker assumes uniform flight valuations \( \rightarrow v_i = 1, \forall i \in F \)

The central decision-maker solves Problems \( P1, P2, \) and \( P3(\rho) \)

Last, we introduce an *Equitable Credit-based Scheduling Mechanism (ECSM)* that enables the
airlines to provide, in addition to the preferred schedule of flights and the set of flight connections to be maintained, the relative scheduling flexibility of their flights through the allocation of non-monetary “credits”. Specifically, each airline \( a \) receives \( v_{|F_a|} \) credits per flight scheduled (i.e., a total of \( v_{|F_a|} \) credits), which it can then allocate to its flights to signal their relative scheduling in flexibility. The more inconvenient, or costly, the rescheduling of any flight \( i \), the more credits are allocated to flight \( i \), subject to the credit-availability constraint \( \sum_{i \in F_a} v_i \leq v_{|F_a|}, \forall a \in A \). These credits are then used by the central decision-maker to optimize scheduling interventions with respect to efficiency, equity, and on-time performance objectives by solving Problems \( P1, P2, \) and \( P3(\rho) \).

### Mechanism 3 Equitable Credit-based Scheduling Mechanism (ESM)

- **The airlines** provide their preferred schedules of flights \( \rightarrow F, F_{\text{arr}}, F_{\text{dep}}, S_{\text{arr}}, S_{\text{dep}} \)
- **The airlines** provide the network connections to be maintained \( \rightarrow C, t^\text{min}, t^\text{max} \)
- **The airlines** allocate their credits to their individual flights \( \rightarrow v, \forall i \in F \), subject to the constraint: \( \sum_{i \in F_a} v_i \leq v_{|F_a|}, \forall a \in A \)
- **The central decision-maker** solves Problems \( P1, P2, \) and \( P3(\rho) \)

Note that the proposed mechanisms introduce an increasing degree of distribution. While the BSM relies on flight scheduling data exclusively and optimizes scheduling interventions from a centralized, efficiency-based perspective, the ESM introduces equity considerations that strive for better distribution of the rescheduling inconvenience across airlines and the ECSM incorporates distributed airline preferences regarding the rescheduling of different flights.

We assume in this paper that the airlines provide their scheduling inputs truthfully and we do not consider explicitly the potential for strategic behaviors in these mechanisms. In fact, strategic behaviors could arise in the submission of flight schedules and network connections and in allocation of credits to the flights. But the ways in which they would manifest themselves are far from obvious and can vary from one case to another. For instance, the mechanisms may create incentives for the airlines to request for scheduling more flights at peak hours than in the absence of scheduling interventions, or to request more connections than needed. But these incentives depend on flights’ scheduling flexibilities, on competing airlines’ schedules, on the scheduling profile at the airport where the scheduling interventions are applied, and on the way scheduling interventions are optimized. Given these complexities, and the limited overall interference of our mechanisms with airline scheduling practices, we ignore these effects in this paper and assume truthful scheduling inputs from the airlines. Moreover, we describe and assess certain aspects of gaming issues, to a limited extent, in Section 6.4.

### 6. Computational Results

We now implement the mechanisms proposed in Section 5 for a case study at JFK Airport. We first describe our experimental setup, and we progressively integrate inter-airline equity objectives
and airline collaboration into scheduling interventions. First, we show that inter-airline equity can be significantly improved under the Equitable Scheduling Mechanism (ESM) at minimal efficiency losses (if any). We then show that accounting for differentiated flight valuations, revealed through credit allocation, improves the outcome of scheduling interventions under the Equitable Credit-based Scheduling Mechanism (ECSM).

6.1. Experimental Setup

We consider data from September 18, 2007 at the John F. Kennedy Airport (JFK). JFK was chosen as the study airport because it is one of the most congested airports in the US, and its peaked schedule of flights offers opportunities for potential delay reductions through scheduling interventions. The day of 09/18/2007 was chosen because no scheduling interventions were in place at JFK in 2007, and because the number of flights scheduled on 09/18 corresponds to the median of the number of daily flights at JFK in 2007. Estimates of JFK’s capacity in various operating conditions were obtained from Simaiakis (2012). Flight schedules were obtained from the Aviation System Performance Metrics (ASPM) database (Federal Aviation Administration, 2013). We group partner airlines together, as major airlines typically coordinate planning and scheduling decisions with their subsidiaries, and passengers can connect between flights operated by partner airlines. Specifically, we consider four groups of airlines at JFK: (i) Delta Airlines (DAL) and its regional partners (which operated a total of 320 flights on 09/18/2007), (ii) American Airlines (AAL) and its regional partners (260 flights), (iii) JetBlue Airways (JBU) (174 flights), and (iv) all other airlines, each of which represents a smaller share of traffic at JFK (408 flights combined). These scheduling data were used to construct $F$, $F_{arr}$, $F_{dep}$, $S_{arr}$, and $S_{dep}$.

We reconstructed aircraft and passenger connections to construct $C$, $t_{\min}$, and $t_{\max}$. Aircraft connections were obtained from the ASPM database. We use the minimum aircraft turnaround time between any pair of flights estimated by Pyrgiotis (2011) as a function of the aircraft type, of the airline and of whether the airport is a hub airport for the airline or not. We use a maximum turnaround time equal to the planned turnaround time plus 15 minutes to maintain comparable aircraft utilization. We obtained passenger connections data from a database developed by Barnhart et al. (2014), based on a discrete choice model for estimating historical passenger flows. We estimate the minimum passenger connection time at JFK as the 5th percentile of the distribution of all planned passenger connection times. Because of data unavailability, we do not reconstruct crew connections here, but their consideration could be easily added to the analysis as and when estimates of historical crew schedules become available Vaze and Wei (2015).

With the 09/18/2007 schedule of flights, the peak expected arrival and departure queue lengths are equal to $\max_{t \in T} E(A_t) = 14.6$ aircraft and $\max_{t \in T} E(D_t) = 28.1$, respectively. We vary the expected arrival queue length target $A_{MAX}$ from 11 to 15 aircraft, and the expected departure queue length target $D_{MAX}$ from 15 to 30 aircraft. These are the targets that can be met under the relatively “mild” process of scheduling interventions considered in this paper and without imposing
a prohibitively large displacement of airline schedules of flights. With these on-time performance targets, the optimal value of the maximum flight displacement $\delta^*$ is equal to 1 period, i.e., all on-time performance targets can be achieved without displacing any flight by more than 15 minutes (Jacquillat and Odoni, accepted for publication, 2015). The results will thus focus on weighted efficiency $\Delta$ (which, for simplicity, we refer to by “efficiency” in the remainder of this section) and on inter-airline equity $\Phi$ (obtained from the vector of airline disutilities, $\sigma$), for any set of on-time performance targets. Unless otherwise specified, we use values of $A_{\text{MAX}} = 11$ aircraft and $D_{\text{MAX}} = 15$ aircraft.

We implemented the Integer Programming models of scheduling interventions in GAMS 24.0 using CPLEX 12.5 and the Dynamic Programming model of capacity utilization and the Stochastic Queuing Model of airport congestion in MATLAB 8.1. We looked for solutions to the Integer Programming models within an optimality gap of 1%. If none was found after 30 minutes, we accepted the solution obtained at that time.

6.2. Inter-Airline Equity: The Equitable Scheduling Mechanism

We compare here the results of the Baseline Scheduling Mechanism (BSM) and the Equitable Scheduling Mechanism (ESM). They both assume uniform flight valuations, but differ in the objective of the scheduling interventions: the BSM maximizes efficiency, under on-time performance targets (Problems $\text{P1}$ and $\text{P2}$), while the ESM, additionally, incorporates inter-airline equity objectives into decision-making (Problems $\text{P3}(\rho^*)$). Comparing the outcomes of these two mechanisms thus shows the extent to which inter-airline equity can be achieved in scheduling interventions under current scheduling conditions and uniform flight valuations.

Note that the solution of Problem $\text{P2}$ is arbitrarily “chosen” by the optimization solver from the set of optimal solutions. In order to characterize the equity range among efficiency-maximizing solutions, we determine the solution which minimizes inter-airline equity, i.e., which lexicographically maximizes airline disutilities, for the optimal value of efficiency. In other words, we characterize the efficiency-maximizing solution that performs the worst in terms of inter-airline equity. We denote this problem by $\text{P2}$.

Table 1 shows the total schedule displacement faced by each airline (which is the same as the number of its flights displaced by 15 minutes each, as the maximum displacement $\delta^*$ is equal to 1 15-minute period), and each airline’s disutility (i.e., its average per-flight displacement) for Problems $\text{P2}$, $\text{P2}$ and $\text{P3}(\rho^*)$. It also reports the ratio of the largest to smallest airline disutility. Note, first, that Problem $\text{P2}$ results in max-min ratios $\max \sigma \over \min \sigma$ ranging between 10 and 50. For the cases considered, AAL and JBU tend to be much more significantly penalized than DAL, which is reflected through more of their flights being rescheduled and through higher disutility values. The set of efficiency-maximizing solutions thus contains highly inequitable outcomes. Problem $\text{P2}$ does not result in the most inequitable outcome in that set, but provides solutions that still impact some airlines (here, AAL, JBU and the “other” airlines) more negatively than others (here, DAL).
Inter-airline equity is achieved only by solving Problem $\overline{P3}(\rho^*)$. In that case, airline disutilities are much closer to each other than by solving Problems $\overline{P2}$ and $P2$. Note that the differences in airlines’ schedules of flights and network connectivities result in all airlines not having the exact same disutility, but differences are very small (i.e., the max-min ratio $\frac{\max_{a} \sigma_a}{\min_{a} \sigma_a}$ is very close to 1) under the equitable solution. Most importantly, the equity-maximizing solution (Problem $\overline{P3}(\rho^*)$) results in the same total displacement as the efficiency-maximizing solution (Problem $\overline{P2}$) in all cases considered. Only the distribution of schedule displacement across the airlines is modified. In other words, efficiency and equity can be jointly maximized and the price of equity is zero (i.e., $\rho^* = 0$).

Table 1: Results of the BSM and ESM: Number of flights displaced and airline disutilities per airline

| On-time targets | Number of flights displaced | Disutility: $\sigma_a = \frac{1}{|F_a|} \sum_{i \in F_a} |u_i|$ |
|-----------------|---------------------------|----------------|
| $A_{\text{MAX}}$ | $D_{\text{MAX}}$ | Model | DAL | AAL | JBU | Others | All | $\sigma_{\text{max}}$ | $\sigma_{\text{min}}$ |
| 14 | 23 | $\overline{P2}$ | 1 | 13 | 1 | 5 | 20 | 0.3% | 5.0% | 0.6% | 1.2% | 16.00 |
| 13 | 20 | $\overline{P2}$ | 1 | 9 | 2 | 8 | 20 | 0.3% | 3.5% | 1.1% | 2.0% | 11.08 |
| 13 | 20 | $\overline{P3}(\rho^*)$ | 4 | 5 | 3 | 8 | 20 | 1.3% | 1.9% | 1.7% | 2.0% | 1.57 |
| 12 | 18 | $\overline{P2}$ | 1 | 28 | 9 | 7 | 46 | 0.3% | 11.2% | 5.2% | 1.7% | 35.69 |
| 12 | 18 | $\overline{P2}$ | 7 | 18 | 8 | 13 | 46 | 2.2% | 6.9% | 4.6% | 3.2% | 3.16 |
| 12 | 18 | $\overline{P3}(\rho^*)$ | 13 | 10 | 7 | 16 | 46 | 4.1% | 3.8% | 4.0% | 3.9% | 1.06 |
| 11 | 15 | $\overline{P2}$ | 1 | 27 | 10 | 18 | 65 | 0.3% | 10.8% | 15.5% | 2.2% | 49.66 |
| 11 | 15 | $\overline{P2}$ | 10 | 27 | 10 | 18 | 65 | 3.1% | 10.4% | 5.7% | 4.4% | 3.32 |
| 11 | 15 | $\overline{P3}(\rho^*)$ | 18 | 14 | 10 | 23 | 65 | 5.6% | 5.4% | 5.7% | 5.6% | 1.07 |
| 11 | 15 | $\overline{P2}$ | 37 | 113 | 39 | 17 | 206 | 11.6% | 43.5% | 22.4% | 4.2% | 10.43 |
| 11 | 15 | $\overline{P2}$ | 50 | 57 | 32 | 67 | 206 | 15.6% | 21.9% | 18.4% | 16.4% | 1.40 |
| 11 | 15 | $\overline{P3}(\rho^*)$ | 57 | 46 | 31 | 72 | 206 | 17.8% | 17.7% | 17.8% | 17.6% | 1.01 |

Therefore, joint optimization of efficiency and equity seems to be achievable under current schedules of flights and uniform flight valuations. In light of the results from Section 4, this suggests that inter-airline variations in flight schedules and network connectivities are relatively weak and do not create, by themselves, a trade-off between efficiency and equity. This is due to the fact that peak-hour schedules typically include flights from several airlines and the schedules of all airlines exhibit network connections to some extent (so the situations depicted in Figures 3 and 4 are not typical of actual scheduling patterns at busy airports). Under these conditions, incorporating inter-airline equity objectives in scheduling interventions can thus yield significant benefits by balancing scheduling adjustments more fairly among the airlines at no efficiency losses.

6.3. Credit Allocation: The Equitable Credit-based Scheduling Mechanism

Under the Equitable Credit-based Scheduling Mechanism (ECSM), the airlines provide, in addition to their preferred schedule of flights and their network connections, the relative scheduling
flexibility of their flights through credit allocation. Without loss of generality, we set \( \bar{v} \) equal to 1, so each airline \( a \) receives \( |\mathcal{F}_a| \) credits. Since the flight scheduling flexibility is private information to the airlines and challenging to estimate using available data, we first simulate the number of credits allocated by the airlines to their different flights, and then incorporate them into our scheduling interventions framework. In this section, we first describe our simulation strategy and then our computational results.

We vary flight valuations for one airline at a time, i.e., we assume that all airlines allocate their credits uniformly across all their flights (i.e., each flight receives exactly \( \bar{v} = 1 \) credit), except one airline \( a \). We partition its set of flights \( \mathcal{F}_a \) in two subsets \( \mathcal{F}_a^{(1)} \) and \( \mathcal{F}_a^{(2)} \) such that \( \mathcal{F}_a^{(1)} \cap \mathcal{F}_a^{(2)} = \emptyset \) and \( \mathcal{F}_a^{(1)} \cup \mathcal{F}_a^{(2)} = \mathcal{F}_a \). We can think of \( \mathcal{F}_a^{(1)} \) (resp. \( \mathcal{F}_a^{(2)} \)) as the set of the most flexible flights (resp. the least flexible flights) of airline \( a \). We choose to represent the valuations of the flights in \( \mathcal{F}_a^{(1)} \) (resp. \( \mathcal{F}_a^{(2)} \)) by a Gamma distribution \( \Gamma_1(\mu_1, k) \) (resp. \( \Gamma_2(\mu_2, k) \)) with mean \( \mu_1 \) (resp. \( \mu_2 \)) and shape parameter \( k \), with \( \mu_1 < \mu_2 \). We adjust the shape parameter of these distributions such that the 95\(^{th} \) percentile of the former distribution coincides with the 5\(^{th} \) percentile of the latter. These choices of distribution and parameters are made in order to provide a transparent and flexible bimodal characterization of positive flight valuations such that the valuations of flights in \( \mathcal{F}_a^{(1)} \) are, in most cases, lower than the valuations of flights in \( \mathcal{F}_a^{(2)} \). Finally, we set the values of flights in \( \mathcal{F}_a^{(1)} \) (resp. \( \mathcal{F}_a^{(2)} \)) equal to \( \Theta_1^{-1}\left(\frac{1}{(|\mathcal{F}_a^{(1)}|+1)}\right) \), \( \Theta_1^{-1}\left(\frac{2}{(|\mathcal{F}_a^{(1)}|+1)}\right) \), ..., \( \Theta_1^{-1}\left(\frac{|\mathcal{F}_a^{(1)}|}{(|\mathcal{F}_a^{(1)}|+1)}\right) \) (resp. \( \Theta_2^{-1}\left(\frac{1}{(|\mathcal{F}_a^{(2)}|+1)}\right) \), \( \Theta_2^{-1}\left(\frac{2}{(|\mathcal{F}_a^{(2)}|+1)}\right) \), ..., \( \Theta_2^{-1}\left(\frac{|\mathcal{F}_a^{(2)}|}{(|\mathcal{F}_a^{(2)}|+1)}\right) \), where \( \Theta_1 \) (resp. \( \Theta_2 \)) denotes the cumulative distribution function of \( \Gamma_1(\mu_1, k) \) (resp. \( \Gamma_2(\mu_2, k) \)). This sampling strategy ensures that the resulting set of flight valuations is distributed “smoothly” across the distributions considered without sampling these values multiple times. For each airline, we vary two parameters:

(i) the fraction of flights in \( \mathcal{F}_a^{(1)} \), denoted by \( \eta = \frac{|\mathcal{F}_a^{(1)}|}{|\mathcal{F}_a|} \) (so that \( 1 - \eta = \frac{|\mathcal{F}_a^{(2)}|}{|\mathcal{F}_a|} \)), and (ii) the mean valuations of flights in \( \mathcal{F}_a^{(1)} \), i.e., \( \mu_1 \) (such that \( \eta \mu_1 + (1 - \eta) \mu_2 = \bar{v} \)). We sort flights from the least valuable to the most valuable using 10 random permutations. In other words, the 10 tests have the same sets of flight valuations, but differ in terms of which flights are more and less flexible.

Table 2 shows ECSM results (with \( A_{\text{MAX}} = 11 \) and \( D_{\text{MAX}} = 15 \)) under different sets of flight valuations provided by DAL (left) and AAL (right)—similar results are obtained by varying the flight valuations provided by the other airlines. In the top half, we assume that \( \mathcal{F}_a^{(1)} \) and \( \mathcal{F}_a^{(2)} \) both comprise 50\% of the flights from DAL or AAL, and we progressively increase the valuation differential \( \mu_2 - \mu_1 \). In the bottom half, we fix \( \mu_1 = 0.75 \) and we progressively decrease the proportion of flights in \( \mathcal{F}_a^{(1)} \) (and we thus increase \( \mu_2 \) to ensure that \( \eta \mu_1 + (1 - \eta) \mu_2 = 1 \)). The table reports, in each scenario, the average schedule displacement \( \sum_{i \in \mathcal{F}_a} |u_i| \) of each airline \( a \) obtained in the equity-maximizing scenario (i.e., Problem \( \mathcal{P}_3(\rho^*) \)), as well as the average prices of equity and efficiency, taken across all 10 samples. The observations from variations in \( \mu_2 - \mu_1 \) (top of the table) and in \( \eta \) (bottom of the table) are threefold. First, as an airline’s flight valuations
become more differentiated, the displacement of this airline’s schedule increases. In turn, each airline faces a trade-off between selecting which flights get rescheduled, on the one hand, and minimizing its overall displacement, on the other. Second, as the variance in any airline’s flight valuations increases, other airlines’ displacements do not change significantly (if anything, they seem to decrease). In other words, the model can account for any airline’s scheduling preferences without negatively impacting the other airlines. Third, the price of equity seems, in most cases, much smaller than the price of efficiency. This suggests that inter-airline equity can be achieved at comparatively small efficiency losses. This is consistent with the theoretical bounds obtained with general utility functions by (Bertsimas et al., 2011a, 2012), and motivates attempts to achieve equity improvements in scheduling interventions.

Table 2: ECSM results: Average displacement ($\sum_{i \in F_a} |u_i|$), price of equity and price of efficiency

<table>
<thead>
<tr>
<th>Scenario</th>
<th>For variations in DAL’s flight valuations</th>
<th>For variations in AAL’s flight valuations</th>
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<tr>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\eta$</td>
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<tr>
<td>0.9</td>
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<td>50%</td>
</tr>
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<td>1.4</td>
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<td>1.5</td>
<td>50%</td>
</tr>
<tr>
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<td>50%</td>
</tr>
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</table>

Finally, Figure 6 shows the disutilities of each airline (i.e., $\sigma_a = \frac{1}{|F_a|} \sum_{i \in F_a} |u_i| = u_i$) in the scenarios shown in Table 2. The full bars correspond to the results of Problem $P3(\rho^*)$ when flight valuations data $v$ are considered, and the empty bars correspond to the results of Problem $P3(\rho^*)$ when flight valuations data $v$ are ignored (i.e., $v_i = 1, \forall i$). As expected, in all cases, the full bars are shorter than the empty bars, indicating that accounting for the flight valuations provided by any airline reduces its disutility. In other words, prioritizing its most flexible flights in the scheduling interventions offsets the potential increase in its overall displacement shown in Table 2. Equally importantly, accounting for flight valuations provided by any airline reduces the disutilities of the
other airlines as well, which is consistent with results shown in Table 2. Across the 36 scenarios considered in Figure 6, we find that taking flight valuations into account led to a reduction of the total weighted displacement by 0.9% to 23.3%, with a mean of 6.5%, and to an improvement in airline utilities by 3.2% to 28.7%, with a mean of 9.34%. In sum, integrating flight valuations in the optimization of scheduling interventions can better satisfy all airlines’ scheduling preferences, thus improving the outcome of the scheduling interventions.

![Variations in $\mu_1$ for DAL ($\eta = 50\%$)](image)

![Variations in $\mu_1$ for AAL ($\eta = 50\%$)](image)

![Variations in $\eta$ for DAL ($\mu_1 = 0.75$)](image)

![Variations in $\eta$ for AAL ($\mu_1 = 0.75$)](image)

Figure 6: Airline utilities when flight valuations are considered (in full) vs. ignored

### 6.4. Discussion

Computational results presented in this section have shown that:

i. Inter-airline equity can be achieved at no, or small, losses in efficiency.

ii. Accounting for flight valuations can improve the outcome of scheduling interventions for all airlines.

In summary, the Equitable Credit-based Scheduling Mechanism (ECSM) optimizes scheduling interventions to attain optimal equity at minimal efficiency losses. It is relatively easy to implement
because it is non-monetary, and considers only marginal scheduling adjustments that are consistent with the current practice at US airports. It is flexible as it does not rely on pre-determined rigid schedule limits, but considers instead airline scheduling preferences as its starting point. It is equitable as it does not prioritize flights from any airline over those of others, and balances the scheduling adjustments fairly among the airlines. It is collaborative as it enables the airlines to provide their preferred schedules of flights, the flight connections to be maintained, and the relative scheduling flexibility of their flights in a simple and transparent way, and it accounts for these preferences in scheduling interventions. In turn, the ECSM optimizes scheduling interventions based on system performance objectives (congestion mitigation) and on distributed airline objectives (satisfying airline scheduling preferences).

Finally, a remaining question is the robustness of the ECSM to airline strategic behaviors. Even though we have assumed in this paper perfect information on the scheduling inputs, no obvious opportunity for strategic behavior appears from our computational results. Most notably, we have seen that non-uniform allocations of credits to flights by any airline tend to increase the displacement faced by that airline, compared to the case where all flights are equally valued, and does not impact the displacement faced by the other airlines significantly. Similarly, we found in other computational experiments (not reported here for paper length considerations) that variations in the number of network connections requested by any airline affect only marginally the displacement faced by any airline. These results suggest that the airlines do not have clear incentives to provide untruthful credit allocations or network connections to decrease their own schedule displacement, or increase the one faced by other airlines. Instead, the ECSM can be used by the airlines to provide their preferences regarding which flights to reschedule and which connections to be maintained.

7. Conclusion

Any airport demand management scheme involves a trade-off between mitigating airport congestion, on the one hand, and minimizing interference with airline competitive scheduling, on the other. In this paper, we have considered a process for airport scheduling interventions under which the airlines provide their scheduling preferences to a central decision-maker, who may then suggest some scheduling adjustments to reduce anticipated delays. We have developed, optimized and assessed mechanisms for such airport scheduling interventions that, for the first time, incorporate inter-airline equity considerations and enable airline collaboration through the sharing of the relative scheduling flexibility of the different flights. These mechanisms rely on an original lexicographic modeling architecture that integrates on-time performance objectives into the decision-making framework for scheduling interventions, and optimizes scheduling interventions based on efficiency and inter-airline equity objectives.

Theoretical and computational results have suggested, first, that under the standard paradigm that “a flight is a flight” (i.e., all flights are equally inconvenient to reschedule), inter-airline equity
can be achieved at no (or minimal) efficiency losses. Moreover, the proposed mechanism can account for the relative scheduling flexibility of flights in determining which flights to move, and which flights’ schedules to leave unchanged. In summary, the outcome of airport scheduling interventions could be improved by (i) ensuring inter-airline equity, and (ii) enabling the airline community to provide their scheduling preferences.

From a practical standpoint, these results suggest that significant benefits could be achieved through the creation of a collaborative environment between the airline community and the central decision-makers in charge of designing scheduling interventions (e.g., the Federal Aviation Administration (FAA) in the US or the schedule coordinators at slot-controlled airports). Under the proposed environment, the airlines would provide their scheduling preferences, the network connections to be maintained, and the relative schedule flexibility of their flights. The central decision-maker would then optimize scheduling interventions, based on efficiency, equity, and on-time performance objectives. Such a collaborative environment would produce delay-mitigating scheduling adjustments that are more consistent with airlines’ scheduling preferences than achieved otherwise. The modeling architecture developed in this paper provides a flexible methodology that could also be extended to account for additional practical considerations or policy requirements, e.g., slot grandfathering, pool of new entrants, secondary trading, etc.

The potential benefits of scheduling interventions also motivate future research directions that have been left out of scope of this paper. First, this paper has relied on simple strategies to simulate airlines’ scheduling inputs into the proposed mechanisms. In future work, this limitation could be addressed by integrating models of airline scheduling into the framework developed here. Second, such models of airlines’ profit-maximizing behaviors would also create an opportunity to analyze strategic interactions among the airlines and the potential for gaming under the proposed scheduling intervention mechanisms. Finally, while this paper has focused on non-monetary mechanisms, market-based mechanisms based on congestion pricing and slot auctions have been proposed in the literature as potential welfare-enhancing mechanisms. The modeling approach developed in this paper could be extended to analyze and compare these alternative mechanisms, based on common efficiency, inter-airline equity, and on-time performance objectives. In turn, the integrated framework of airport scheduling interventions and airport operations provides the methodological foundation to systematically address problems of airport capacity allocation to mitigate delay externalities, promote airline competition, and maximize social welfare.

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Appendix

In this Appendix, we characterize the optimal solution of Problem $P(\Delta)$ defined as follows, where $\Delta$ designates any non-negative integer. Problem $P(\Delta)$ involves allocating a “budget” of $\Delta$ items (think of each item as an inconvenience or cost) across “groups” (or airlines, in our case) indexed by $a \in \mathcal{A}$ in a way that lexicographically minimizes the weighted cost borne by any group, where the weight for each group $a$ is given by $\frac{1}{|F_a|}$.

$$\text{lex min} \left( \frac{U_a}{|F_a|} \right)_{a \in \mathcal{A}} \quad \text{s.t.} \quad \sum_{a \in \mathcal{A}} U_a \geq \Delta \quad U_a \geq 0, U_a \text{ integer}$$

We introduce the following notations. We denote the indicator function by $1$. For each solution vector $(U_a)_{a \in \mathcal{A}}$, we sort $\left( \frac{U_a}{|F_a|} \right)_{a \in \mathcal{A}}$ by non-increasing order, and we denote by $s_i(U)$ the $i$-th element of the resulting vector. In other words, $s_1(U) = \max_{a \in \mathcal{A}} \frac{U_a}{|F_a|}$. Recursively, if $a_1, \ldots, a_{i-1}$ are such that $\frac{U_{a_j}}{|F_{a_j}|} = s_j(U), \forall j = 1, \ldots, i-1$, then $s_i(U) = \max_{a \notin \{a_1, \ldots, a_{i-1}\}} \frac{U_a}{|F_a|}$. We denote by $\Theta(U)$ the set of indices that attain the maximum of $\left( \frac{U_a}{|F_a|} \right)_{a \in \mathcal{A}}$, i.e., $\Theta(U) = \{ a \in \mathcal{A}, \frac{U_a}{|F_a|} = s_1(U) \}$.

Last, we write $U \succeq_{\text{lex}} V$ (resp. $U \succ_{\text{lex}} V$) to signify that $\left( \frac{U_a}{|F_a|} \right)_{a \in \mathcal{A}}$ is lexicographically larger (resp. strictly larger) than $\left( \frac{V_a}{|F_a|} \right)_{a \in \mathcal{A}}$. In other words, $U \succ_{\text{lex}} V$ if $\exists i \in \{1, \ldots, |\mathcal{A}|\}, s_1(U) = s_1(V), s_2(U) = s_2(V), \ldots, s_i-1(U) = s_{i-1}(V)$ and $s_i(U) > s_i(V)$, and $U \succeq_{\text{lex}} V$ if $U \succ_{\text{lex}} V$ or $s_i(U) = s_i(V), \forall i = 1, \ldots, |\mathcal{A}|$.

Lemma 1 shows that, for any optimal solution of $P(\Delta)$, the constraint $\sum_{a \in \mathcal{A}} U_a \geq \Delta$ is binding.

**Lemma 1.** Any optimal solution $(U_a)_{a \in \mathcal{A}}$ of $P(\Delta)$ satisfies $\sum_{a \in \mathcal{A}} U_a = \Delta$.

**Proof.** Let us assume that $(U_a)_{a \in \mathcal{A}}$ is an optimal solution of $P(\Delta)$ such that $\sum_{a \in \mathcal{A}} U_a > \Delta$. We denote by $\varepsilon$ the integer defined by $\varepsilon = \sum_{a \in \mathcal{A}} U_a - \Delta > 0$. We introduce a vector $(\eta_a)_{a \in \mathcal{A}}$ of non-negative integers which satisfies the condition $\sum_{a \in \mathcal{A}} \eta_a = \varepsilon$ (Note that $(\eta_a)_{a \in \mathcal{A}}$ thus defined is not unique, but we can choose any vector that satisfies these properties). We then define $U^*_a = U_a - \eta_a, \forall a \in \mathcal{A}$. By construction, $\sum_{a \in \mathcal{A}} U^*_a = \Delta$, $U^*_a \leq U_a, \forall a \in \mathcal{A}$ and $\exists a \in \mathcal{A}, U^*_a < U_a$. Thus $U \succ_{\text{lex}} U^*$, which contradicts the fact that $(U_a)_{a \in \mathcal{A}}$ is an optimal solution of $P(\Delta)$. \( \square \)

In Lemma 2, we prove an intermediate result that shows that, if we start with an optimal solution of Problem $P(\Delta)$ and we add one item to any group, then the resulting weighted cost borne by that group is at least equal to the optimal value of the largest weighted cost.

**Lemma 2.** If $(U_a)_{a \in \mathcal{A}}$ is an optimal solution of $P(\Delta)$, then $\frac{U_a + 1}{|F_a|} \geq s_1(U), \forall c \in \mathcal{A}$.
\textbf{Proof.} Let us consider }c \in A. \text{ If } \frac{U_b}{|\mathcal{P}_|} = s_1(U), \text{ then } \frac{U_b+1}{|\mathcal{P}_|} > s_1(U). \text{ If } \frac{U_a}{|\mathcal{P}_|} < s_1(U), \text{ then there exists } b \neq c, \frac{U_b}{|\mathcal{P}_|} = s_1(U). \text{ We define } \mathcal{U} \text{ as follows: } \mathcal{U}_c = U_c + 1, \mathcal{U}_b = U_b - 1 \text{ and } \mathcal{U}_a = U_a, \forall a \neq b, c. \text{ If } |\Theta(U)| > 1, \text{ then } U_a = \mathcal{U}_a, \forall a \in \Theta(U) \setminus \{b\} \text{ and } s_1(U) = s_2(U) = \ldots = s_{|\Theta(U)|-1}(U) = s_1(\mathcal{U}) = \ldots = s_{|\Theta(U)|-1}(\mathcal{U}). \text{ Since } U \text{ is an optimal solution of } \mathcal{P}(\Delta), \text{ then } s_{|\Theta(U)|}(U) \geq s_{|\Theta(U)|}(\mathcal{U}) = s_1(U). \text{ We have: } s_{|\Theta(U)|}(\mathcal{U}) = \max \left\{ \max_{a \in \Theta(U)} \frac{U_a}{|\mathcal{P}_|}, \frac{U_b+1}{|\mathcal{P}_|}, \frac{U_c+1}{|\mathcal{P}_|} \right\}. \text{ We know that } \max_{a \in \Theta(U), a \neq c} \frac{U_a}{|\mathcal{P}_|} < s_1(U) \text{ and that } \frac{U_b+1}{|\mathcal{P}_|} < s_1(U). \text{ This implies that } \frac{U_b+1}{|\mathcal{P}_|} \geq s_1(U). \square 

The first two lemmas show that if the constraint } \sum_{a \in A} U_a \geq \Delta \text{ is not binding, or if there exists } c \in A \text{ such that } \frac{U_b+1}{|\mathcal{P}_|} < s_1(U), \text{ then } U \text{ is not an optimal solution of } \mathcal{P}(\Delta). \text{ In Lemma 3, we identify necessary conditions over a feasible solution } (V_a)_{a \in A} \text{ of Problem } \mathcal{P}(\Delta) \text{ to ensure that } U \succeq_{\text{lex}} V \text{ even if } (U_a)_{a \in A} \text{ does these two verifications.}

\textbf{Lemma 3.} Let } (U_a)_{a \in A} \text{ be such that } \sum_{a \in A} U_a = \Delta \text{ and } \frac{U_b+1}{|\mathcal{P}_|} \geq s_1(U), \forall c \in A, \text{ and } V \text{ denote any feasible solution of } \mathcal{P}(\Delta). \text{ If } U \succeq_{\text{lex}} V, \text{ then there exist } b_1, b_2, c_1, \ldots, c_q \in A \text{ such that: (i) } U_{b_1} = s_1(U), \forall i = 1, \ldots, q, \text{ (ii) } U_{b_i+1} = s_1(U), \forall i = 1, \ldots, q, \text{ (iii) } V_{b_i} = U_{b_i} - 1, \forall i = 1, \ldots, q, \text{ (iv) } V_{c_1} = U_{c_1} + 1, \forall i = 1, \ldots, q, \text{ and (v) } V_a = U_a, \forall a \neq b_1, \ldots, b_q, c_1, \ldots, c_q.

\textbf{Proof.} Let } (U_a)_{a \in A} \text{ be a feasible solution of } \mathcal{P}(\Delta) \text{ such that } \sum_{a \in A} U_a = \Delta \text{ and } \frac{U_b+1}{|\mathcal{P}_|} \geq s_1(U), \forall c \in A. \text{ We can partition the set } A \text{ into } \Omega_1 = \left\{ a \in A, \frac{U_a}{|\mathcal{P}_|} = s_1(U) \right\}, \Omega_2 = \left\{ a \in A, \frac{U_a+1}{|\mathcal{P}_|} = s_1(U) \right\} \text{ and } \Omega_3 = \left\{ a \in A, \frac{U_a}{|\mathcal{P}_|} < s_1(U) < \frac{U_a+1}{|\mathcal{P}_|} \right\}. \text{ Let } V \text{ be any feasible solution of } \mathcal{P}(\Delta) \text{ such that } U \succeq_{\text{lex}} V.

First, we note that } V_b \leq U_b, \forall b \in \Omega_1 \cup \Omega_3 \text{ and } V_b \leq U_b + 1, \forall b \in \Omega_2. \text{ Indeed, if } \exists b \in \Omega_1 \cup \Omega_3, V_b > U_b \text{ (i.e., } V_b \geq U_b + 1 \text{ since } V_b \text{ and } U_b \text{ are integer), then } \frac{V_b}{|\mathcal{P}_|} \geq \frac{U_b+1}{|\mathcal{P}_|} \geq s_1(U), \text{ thus } V \succeq_{\text{lex}} U. \text{ Similarly, if } \exists b \in \Omega_2, V_b \geq U_b + 2, \text{ then } \frac{V_b}{|\mathcal{P}_|} \geq \frac{U_b+1}{|\mathcal{P}_|} \geq \frac{U_b+2}{|\mathcal{P}_|} = s_1(U), \text{ thus } V \succeq_{\text{lex}} U. \text{ We denote by } X_1^- = \sum_{a \in \Omega_1} \mathbb{1}(V_a < U_a) \text{ and } X_3^- = \sum_{a \in \Omega_3} \mathbb{1}(V_a < U_a), \text{ and by } X_2^+ = \sum_{a \in \Omega_2} \mathbb{1}(V_a = U_a + 1).

Second, we show that } X_2^+ = X_1^- \text{ and } X_3^- = 0. \text{ If } \exists b \in A, V_b < U_b \text{ (i.e., } V_b \leq U_b - 1), \text{ then } \exists c \in \Omega_2, V_c \geq U_c + 1 \text{ (since } \sum_{a \in A} U_a = \Delta \text{ and } V_a \leq U_a, \forall a \in \Omega_1 \cup \Omega_3). \text{ Therefore, } X_2^+ \geq X_1^- + X_3^- \text{ Moreover, the number of indices such that } \frac{V_a}{|\mathcal{P}_|} = s_1(U) \text{ is equal to: } \sum_{a \in \Omega_1} \mathbb{1}(V_a = U_a) + \sum_{a \in \Omega_2} \mathbb{1}(V_a = U_a + 1) = |\Omega_1| - X_1^- + X_2^+. \text{ Therefore, } s_1(V) = \ldots = s_{|\Omega_1|-X_1^-+X_2^+}(V) = s_1(U) = \ldots = s_{|\Omega_1|}(U), \text{ while } s_{|\Omega_1|+1}(U) < s_1(U). \text{ Thus, if } X_2^+ > X_1^-, \text{ then } V \succeq_{\text{lex}} U, \text{ which contradicts our assumption. Therefore, } X_2^+ = X_1^-, \text{ which also implies that } X_3^- = 0.

Third, we note that } V_b \geq U_b - 1, \forall b \in \Omega_1. \text{ Indeed, if } \exists b_0 \in \Omega_1, V_{b_0} \leq U_{b_0} - 2, \text{ then } \exists c_0 \neq d_0 \in \Omega_2, V_{c_0} = U_{c_0} + 1 \text{ and } V_{d_0} = U_{d_0} + 1, \text{ and for each } b \neq b_0 \in \Omega_1 \text{ such that } V_b < U_b, \text{ there exists } c \neq c_0, d_0 \in \Omega_2 \text{ such that } V_c = U_c + 1. \text{ This results in } X_2^+ \geq X_1^- + 1, \text{ which contradicts } X_2^+ = X_1^-.

In summary, we have shown that, if } U \succeq_{\text{lex}} V, \text{ then } V_b = U_b, \forall b \in \Omega_3 \text{ (since } X_3^- = 0) \text{ and for each } b \in \Omega_1 \text{ such that } V_b < U_b, \text{ there exists exactly one } c \in \Omega_2 \text{ such that } V_c = U_c + 1.
Therefore, there exist \( b_1, \ldots, b_q \in \Omega_1 \) and \( c_1, \ldots, c_q \in \Omega_2 \) such that \( V_{b_j} = U_{b_j} - 1, \forall j = 1, \ldots, q \), \( V_{c_j} = U_{c_j} + 1, \forall j = 1, \ldots, q \), and \( V_a = U_a, \forall a \neq b_1, \ldots, b_q, c_1, \ldots, c_q \).

Lemma 4 then shows that if we start with a solution of Problem \( \mathcal{P}(\Delta) \), then we can construct a solution of Problem \( \mathcal{P}(\Delta - 1) \) by removing one item from the group (or one of the groups) that bears the largest weighted cost.

**Lemma 4.** If \( \Delta \geq 1 \) and \((U_a)_{a \in \mathcal{A}}\) is an optimal solution of \( \mathcal{P}(\Delta) \), then there exists an optimal solution \((U_a^0)_{a \in \mathcal{A}}\) of \( \mathcal{P}(\Delta - 1) \) and \( a_0 \in \mathcal{A} \) such that: \( U_{a_0}^0 = U_{a_0} - 1 \) and \( U_a^0 = U_a, \forall a \neq a_0 \).

**Proof.** Let \((U_a)_{a \in \mathcal{A}}\) be an optimal solution of \( \mathcal{P}(\Delta) \). We choose \( a_0 \in \arg \min_{a \in \Theta(U)} |\mathcal{F}_a| \), and we define \( U^0 \) such that: \( U_{a_0}^0 = U_{a_0} - 1 \) and \( U_a^0 = U_a, \forall a \neq a_0 \). Note that \( U_a^0 \leq U_a, \forall a \in \mathcal{A} \), so \( s_i(U) \geq s_i(U^0), \forall i = 1, \ldots, |\mathcal{A}| \). Let \((V_a)_{a \in \mathcal{A}}\) be any feasible solution of \( \mathcal{P}(\Delta - 1) \) and we need to show that \( V \succeq_{\text{lex}} U^0 \).

First, we have \( \sum_{a \in \mathcal{A}} U_a^0 = \Delta - 1 \). According to Lemma 2, we know that \( \frac{U_{a_0}^0 + 1}{|\mathcal{F}_{a_0}|} \geq s_1(U), \forall a \in \mathcal{A} \), so \( \frac{U_{a_0}^0 + 1}{|\mathcal{F}_{a_0}|} \geq s_1(U^0), \forall a \neq a_0 \). Since \( \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} = s_1(U) \), we also have \( \frac{U_{a_0}^0 + 1}{|\mathcal{F}_{a_0}|} \geq s_1(U^0), \forall a \neq a_0 \). Therefore, \( U^0 \) satisfies the conditions of Lemma 3. Thus, \( V \succeq_{\text{lex}} U^0 \) unless there exist \( b_1, \ldots, b_q, c_1, \ldots, c_q \in \mathcal{A} \) such that: (i) \( \frac{V_{b_i}}{|\mathcal{F}_{b_i}|} = s_1(U^0), \forall i = 1, \ldots, q \), (ii) \( \frac{V_{c_i}}{|\mathcal{F}_{c_i}|} = s_1(U^0), \forall i = 1, \ldots, q \), (iii) \( V_{b_i} = U_{b_i}^0 - 1, \forall i = 1, \ldots, q \), (iv) \( V_{c_i} = U_{c_i}^0 + 1, \forall i = 1, \ldots, q \), and (v) \( V_a = U_a^0, \forall a \neq b_1, \ldots, b_q, c_1, \ldots, c_q \).

We now assume these conditions to be satisfied. By construction, \( \frac{V_{b_i}}{|\mathcal{F}_{b_i}|} = \frac{U_{b_i}^0}{|\mathcal{F}_{b_i}|}, \forall i = 1, \ldots, q \). Therefore, \( V \succeq_{\text{lex}} U^0 \) if and only if \( \frac{V_{b_i}}{|\mathcal{F}_{b_i}|} = \frac{U_{b_i}^0}{|\mathcal{F}_{b_i}|} \) \( \forall i = 1, \ldots, q \).

We now note that if \( \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} = s_1(U^0) \), then \( \frac{U_{a_0}^0 + 1}{|\mathcal{F}_{a_0}|} > s_1(U^0), \forall c \neq a_0 \). We already know from Lemma 2 that \( \frac{U_{a_0}^0 + 1}{|\mathcal{F}_{a_0}|} \geq s_1(U^0), \forall c \neq a_0 \). Let us assume that \( \exists c \in \mathcal{A}, \frac{U_{a_0}^0 + 1}{|\mathcal{F}_{c}|} = \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} = s_1(U^0) \). We can then define \( U \) such that \( U_c = U_c + 1, U_{a_0} = U_{a_0} - 1 = U_{a_0}^0 \) and \( U_a = U_a, \forall a \neq a_0, c \). \( U \) is a feasible solution of \( \mathcal{P}(\Delta) \). Since \( \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} = s_1(U^0) \), \( U_{a_0}^0 \leq \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \), \( \forall a \neq a_0 \). Therefore, \( s_1(U) = \frac{U_{a_0}^0 - 1}{|\mathcal{F}_{a_0}|} < \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} = s_1(U) \). This contradicts the fact that \( U \) is an optimal solution of \( \mathcal{P}(\Delta) \).

Last, we show that for each \( b \) such that \( \frac{U_{b_0}}{|\mathcal{F}_{b}|} = s_1(U^0) \) and for each \( c \) such that \( \frac{U_{a_0}^0 + 1}{|\mathcal{F}_{c}|} = s_1(U^0) \), we have \( \frac{U_{b_0}^0 - 1}{|\mathcal{F}_{b}|} \geq \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \). Based on the previous result, \( b \neq a_0 \) (otherwise there exists no \( c \) such that \( \frac{U_{a_0}^0 + 1}{|\mathcal{F}_{c}|} = s_1(U^0) \)), so we need to consider only the following two cases:

(i) If \( b, c \neq a_0 \), we define \( U \) as follows: \( U_c = U_c + 1, U_b = U_b - 1 \) and \( U_a = U_a, \forall a \neq b, c \).

Since \( \sum_{a \in \mathcal{A}} U_a = \Delta, U \) is a feasible solution of \( \mathcal{P}(\Delta) \), so \( U \succeq_{\text{lex}} U \). Since \( \frac{U_{b_0}}{|\mathcal{F}_{b}|} = \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \) and \( \frac{U_a}{|\mathcal{F}_a|}, \forall a \neq b, c \), this implies that \( \frac{U_{b_0}}{|\mathcal{F}_{b}|} \geq \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \), i.e. \( \frac{U_{b_0} - 1}{|\mathcal{F}_{b}|} \geq \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \), i.e.: \( \frac{U_{b_0} - 1}{|\mathcal{F}_{b}|} \geq \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \).

(ii) If \( c = a_0 \), then \( \frac{U_{b_0}}{|\mathcal{F}_{b}|} = \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \), i.e., \( \frac{U_b}{|\mathcal{F}_b|} = \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \) and \( b \in \Theta(U) \). By construction, \( a_0 \in \arg \min_{a \in \Theta(U)} |\mathcal{F}_a| \), so \( |\mathcal{F}_{a_0}| \leq |\mathcal{F}_b| \). We thus have \( \frac{U_{b_0}}{|\mathcal{F}_{b}|} - \frac{1}{|\mathcal{F}_b|} \geq \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} - \frac{1}{|\mathcal{F}_{a_0}|} \), i.e., \( \frac{U_{b_0} - 1}{|\mathcal{F}_{b}|} \geq \frac{U_{a_0}^0}{|\mathcal{F}_{a_0}|} \).
Therefore, \( \frac{V_{b_i}}{|F_{b_i}|} \geq \frac{U_{c_i}^0}{|F_{c_i}|}, \forall i = 1, ..., q \). This implies that \( V \succeq_{\text{lex}} U^0 \). \( \square \)

Lemma 5 extends Lemma 4 to construct, from a solution of Problem \( P(\Delta) \), solutions of Problems \( P(\Delta - 1), ..., P(0) \) such that each one differs from the following one by only one element.

**Lemma 5.** If \( \Delta \geq 1 \) and \( (U_a^\Delta)_{a \in A} \) is an optimal solution of \( P(\Delta) \), then there exist \( (U_a^{\Delta-1})_{a \in A}, ..., (U_a^0)_{a \in A} \) that are optimal solutions of \( P(\Delta - 1), ..., P(0) \), respectively, and \( a_1, ..., a_\Delta \in A \) such that:

\[
U_{a_i}^{\Delta-1} = U_{a_i}^i - 1 \quad \text{and} \quad U_{a_i}^i = U_{a_i}^i, \forall a \neq a_i, \forall i = 1, ..., \Delta.
\]

**Proof.** Proof. This is obtained directly by repeatedly applying Lemma 4 to \( P(\Delta) \), \( P(\Delta - 1), ..., P(2) \) and \( P(1) \).

We now introduce the following notations. We denote by \( \gamma \) the greatest common divisor of \( (|F_a|)_{a \in A^\prime} \), i.e., \( \gamma = \gcd (|F_a|)_{a \in A^\prime} \). We also introduce \( \xi_a = \frac{|F_a|}{\gamma} \) and \( N = \sum_{a \in A} \xi_a \). We show in Lemma 6 that if we know a solution of Problem \( P(\Delta) \), then we can construct easily the solution of \( P(\Delta + N) \) by adding \( \xi_a \) items to each group \( a \).

**Lemma 6.** If \( (U_a)_{a \in A} \) is an optimal solution of \( P(\Delta) \), then \( (U_a + \xi_a)_{a \in A} \) is an optimal solution of \( P(\Delta + N) \).

**Proof.** Proof. The construction of the proof is very similar to that of Lemma 4. Let \( (U_a)_{a \in A} \) be an optimal solution of \( P(\Delta) \). We have:

\[
\frac{U_a + \xi_a}{|F_a|} = \frac{1}{\gamma} + \frac{U_a}{|F_a|}, \forall a \in A, \text{ and thus: } s_i(U + \xi) = \frac{1}{\gamma} + s_i(U), \forall i = 1, ..., |A|.
\]

Let \( (V_a)_{a \in A} \) be any feasible solution of \( P(\Delta + N) \) and we need to show that \( V \succeq_{\text{lex}} U + \xi \).

First, we have \( \sum_{a \in A} (U_a + \xi_a) = \Delta + N \). According to Lemma 2, we know that \( \frac{U_a + \xi_a}{|F_a|} \geq s_1(U), \forall a \in A \), so \( \sum_{a \in A} (U_a + \xi_a) \geq \sum_{a \in A} s_1(U) = \Delta + N \), \forall a \in A \). Therefore, \( U + \xi \) satisfies the conditions of Lemma 3.

Thus, \( V \succeq_{\text{lex}} U + \xi \) unless there exist \( b_1, ..., b_q, c_1, ..., c_q \in A \) such that:

(i) \( \frac{V_{b_1} + \xi_{b_1}}{|F_{b_1}|} = s_1(U + \xi), \forall i = 1, ..., q \),

(ii) \( \frac{U_{a_i} + \xi_{a_i} + 1}{|F_{a_i}|} = s_1(U + \xi), \forall i = 1, ..., q \),

(iii) \( V_{c_i} = U_{b_i} + \xi_{b_i} - 1, \forall i = 1, ..., q \),

(iv) \( V_{c_i} = U_{a_i} + \xi_{a_i} + 1, \forall i = 1, ..., q \),

and (v) \( V_a = U_a + \xi_a, \forall a \neq b_1, ..., b_q, c_1, ..., c_q \). We now assume these conditions to be satisfied. By construction, \( \frac{V_{b_i}}{|F_{b_i}|} = \frac{U_{b_i} + \xi_{b_i}}{|F_{b_i}|}, \forall i = 1, ..., q \). Therefore, \( V \succeq_{\text{lex}} U + \xi \) if and only if

\[
\left( \frac{V_{b_i}}{|F_{b_i}|} \right)_{i=1,...,q} \succeq_{\text{lex}} \left( \frac{U_{c_i} + \xi_{c_i}}{|F_{c_i}|} \right)_{i=1,...,q}.
\]

We now show that for each \( b \) such that \( V_{b_i} + \xi_{b_i} = s_1(U + \xi) \) and for each \( c \) such that \( U_{c_i} + \xi_{c_i} + 1 = s_1(U + \xi) \), we have \( \frac{U_{b_i} + \xi_{b_i} - 1}{|F_{b_i}|} \geq \frac{U_{c_i} + \xi_{c_i} + 1}{|F_{c_i}|} \). We define \( \bar{U} \) as follows: \( \bar{U}_c = U_c + 1, \bar{U}_b = U_b - 1 \) and \( \bar{U}_a = U_a, \forall a \neq b, c \). Since \( \sum_{a \in A} \bar{U}_a = \Delta \), \( \bar{U} \) is a feasible solution of \( P(\Delta) \), so \( \bar{U} \succeq_{\text{lex}} U \). Since \( \frac{V_{b_i}}{|F_{b_i}|} = \frac{U_{b_i}}{|F_{b_i}|} \) and \( \frac{V_{c_i}}{|F_{c_i}|} = \frac{U_{c_i}}{|F_{c_i}|}, \forall a \neq b, c \), this implies that \( \frac{V_{b_i}}{|F_{b_i}|} \geq \frac{U_{c_i}}{|F_{c_i}|} \), i.e., \( \frac{U_{b_i} + \xi_{b_i} - 1}{|F_{b_i}|} \geq \frac{U_{c_i} + \xi_{c_i} + 1}{|F_{c_i}|} \), i.e., \( \frac{U_{b_i} + \xi_{b_i} - 1}{|F_{b_i}|} \geq \frac{U_{c_i} + \xi_{c_i} + 1}{|F_{c_i}|} \).

Therefore, \( \frac{V_{b_i}}{|F_{b_i}|} \geq \frac{U_{b_i} + \xi_{b_i}}{|F_{b_i}|}, \forall i = 1, ..., q \). This implies that \( V \succeq_{\text{lex}} U + \xi \). \( \square \)

Finally, Lemma 7 uses the result from Lemma 6 to construct a sequence of \( N \) elements \( a_1, ..., a_N \) in \( A \) that contains exactly \( \xi_a \) repetitions of each \( a \in A \) and from which we can construct the solution of Problem \( P(\Delta) \), for any \( \Delta \geq 0 \). Their order is chosen such that, for each \( \Delta = 1, ..., N \), we can
construct a solution of Problem $\mathcal{P}(\Delta)$ by counting the number of times that $a_i$ is equal to $a$, for $i = 1, ..., \Delta$. In other words, $\mathcal{P}(\Delta)$ is solved by the vector $U$ defined by $U_a = \sum_{i=1}^{\Delta} \mathbb{1}(a_i = a), \forall a \in A$. If $\Delta > N$, then we use a similar process based on the Euclidean division of $\Delta$ by $N$ in combination with Lemma 6.

**Lemma 7.** There exists a sequence $(a_1, ..., a_N) \in \mathcal{A}^N$ such that, for any $\Delta \geq 0$, if $q$ and $r$ denote the quotient and remainder of the Euclidean division of $\Delta$ by $N$, then $(U_a)_{a \in A}$ defined by $U_a = q\xi_a + \sum_{i=1}^r \mathbb{1}(a_i = a), \forall a \in A$ is an optimal solution of $\mathcal{P}(\Delta)$. (By convention, $\sum_{i=1}^0 \mathbb{1}(a_i = a) = 0, \forall a \in A$.)

**Proof.** We consider an optimal solution $(U_a^N)_{a \in A}$ of Problem $\mathcal{P}(N)$. According to Lemma 5, there exist $(U_a^0)_{a \in A}$, ..., $(U_a^{N-1})_{a \in A}$ and $a_1, ..., a_N \in A$ such that for all $p = 1, ..., N$: $(U_a^p)_{a \in A}$ is an optimal solution of $\mathcal{P}(p)$ and $U_a^{p-1} = U_a^p - 1$ and $U_a^0 = U_a^p, \forall a \neq a_p$. In other words, there exists a sequence $a_1, ..., a_N$ such that for all $p = 1, ..., N - 1$, the vector $U$ defined by $U_a = \sum_{i=1}^p \mathbb{1}(a_i = a)$ solves Problem $\mathcal{P}(p)$. We apply this result to $r \in \{0, ..., N - 1\}$, and according Lemma 6 (applied $q$ times), the vector $U$ defined by $U_a = q\xi_a + \sum_{i=1}^r \mathbb{1}(a_i = a), \forall a \in A$ is an optimal solution of $\mathcal{P}(r + qN)$, i.e., of $\mathcal{P}(\Delta)$. \hfill $\square$

We end with a simple example to illustrate how the solution of $\mathcal{P}(\Delta)$ is constructed. We consider a case with three “groups” (i.e., $|\mathcal{A}| = 3$) such that $|\mathcal{F}_1| = 20$, $|\mathcal{F}_2| = 30$ and $|\mathcal{F}_3| = 50$. We have $\gamma = 10$, $\xi_1 = 2$, $\xi_2 = 3$, $\xi_3 = 5$ and $N = 10$. We construct the sequence $(a_1, ..., a_N)$ by sorting in the increasing order the elements in the following set $\left\{ \frac{1}{5}, \frac{2}{5}, ..., \frac{6}{5}, \frac{1}{2}, \frac{2}{2}, ..., \frac{6}{2}, \frac{1}{3}, \frac{2}{3}, ..., \frac{6}{3} \right\}$, i.e., in the set $\left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{5}, \frac{2}{5}, ..., \frac{3}{5}, 1 \right\}$. The sorted set is $\left\{ \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, ..., \frac{4}{3}, 1, 1, 1 \right\}$. By taking the corresponding indices, we obtain the following sequence: $(a_1, ..., a_N) = (3, 2, 3, 1, 3, 2, 3, 3, 2, 1)$. In other words, we allocate the first item (if $\Delta = 1$) to $a = 3$ (with a corresponding objective function value equal to $\frac{1}{3}$), the second item (if $\Delta = 2$) to $a = 2$ (with a corresponding objective function value equal to $\frac{1}{2}$), the third item (if $\Delta = 3$) to $a = 3$ (with a corresponding objective function value equal to $\frac{1}{3}$), etc. Note that the last three items (with $\Delta = 8, 9, 10$) are allocated to $a = 3, a = 2$ and $a = 1$ in this specific sequence to guarantee that the sequence lexicographically minimizes the elements in the set for $\Delta = 8$ and $\Delta = 9$. Table 3 shows a vector $(U_a)_{a \in A}$ that solves Problem $\mathcal{P}(\Delta)$ for different values of $\Delta$, based on the sequence $(a_1, ..., a_N)$ thus constructed.

| $\Delta$ | $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | ...
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| 3         | 1   | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 | 10 | 10 |...

Table 3: Solution of $\mathcal{P}(\Delta)$ for $|\mathcal{A}| = 3$, $\xi_1 = 2$, $\xi_2 = 3$, $\xi_3 = 5$, and different values of $\Delta$. 

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References


